OBTAiNABLE ACCURACY IN THE SOLUTION OF PRACTICAL PROBLEMS BY SMALL-AMPLITUDE WAVE THEORY

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Zhurnal prikladnoi mekhaniki i tekhnicheskoii fiziki, No. 1, pp. 88-87, 1965

Results obtainable using the theory of progressive waves of small amplitude and certain fundamental solutions relating to waves of finite height are investigated. The theoretical findings are compared with existing experimental data. It is established that the best agreement between the theoretical and experimental profiles of a plane wave is achieved with constructions based on Kozhevnikov's graphs and the equations of motion in the second approximation with respect to the wave height in the form proposed by Mich. The limits within which it is expedient to use the theoretical formulas of the theory of small-amplitude waves and the theory of the second approximation with respect to the wave height are found and proved for the particle velocity, excess pressure, energy flux, and the energy of a single wave.

In solving practical problems connected with wave motion at the surface of a heavy incompressible liquid and its effect on obstacles, it is customary to use the theory of potential waves of infinitely small amplitude. For this form of motion, in the case of progressive waves in water of finite depth, the projections of the velocity on the coordinate axes and the variable part of the local pressure are given by the formulas [3, 4]:

\[ v_x = \frac{2h}{\sinh kH} \frac{h \sin (k(H + z))}{\sinh kH} \cos (st - kx), \]
\[ v_z = \frac{2h}{\sinh kH} \frac{h \sin (k(H + z))}{\sinh kH} \sin (st - kx), \]
\[ p = \frac{2gh}{\sinh kH} \frac{h \sin (k(H + z))}{\sinh kH} \cos (st - kx). \]

where \( h \) is the height of the wave, \( \lambda \) is the wavelength, \( \tau \) is the period of the wave, \( H \) is the depth of the water, \( \rho \) is the density of the water, \( g \) is the acceleration of gravity, \( z \) the vertical coordinate, taken with a minus sign below the static level, and \( x \) the horizontal coordinate at the level of the static horizon.

The linear theory of small-amplitude waves is simple and very convenient for practical purposes. However, in view of the assumptions made, it appears necessary to make a more accurate determination of its limits of applicability and at the same time to show what solutions are to be preferred when the theory fails to give the required accuracy.

It is known that one of the principal indicators of correspondence between wave theory and the phenomenon in question is good agreement between the theoretical and the actual profiles of the agitated surface. In order to make such a comparison, in a wave tank measuring 40 \( \times \) 1.0 \( \times \) 1.2 m with glazed side walls we performed a series of experiments to record wave profiles on still and motion-picture film. In addition, we constructed profiles of waves of trochoidal form and small amplitude from Kozhevnikov's data and from the following relations:

1. Stokes [5]:

\[ \eta_0 = \frac{h}{2} \cos \kappa x - \frac{kh^2}{16} \frac{ch kH}{\sinh^3 kH} (ch 2kH + 2) \cos 2kx. \]  

2. Nekrasov [6]:

\[ x = -\frac{\lambda}{2\pi} \theta - \frac{h}{2} \sinh \frac{2\pi H}{\lambda} \sin \theta, \quad z = \frac{h}{2} \cos \theta. \]  

3. Mich [2]:

\[ x = x_0 + \frac{h}{2} \frac{ch k(H + z_0)}{\sinh kH} \sin (st - kx_0) - \frac{kh^2}{16} \frac{sh 2(\tau t - kx_0)}{\sinh^3 kH} \left[ 1 - \frac{3}{2} \frac{ch 2k(H + z_0)}{\sinh^2 kH} \right] \]
\[ z = z_0 + \frac{h}{2} \frac{sh k(H + z_0)}{\sinh kH} \cos (st - kx_0) + \frac{kh^2}{16} \frac{sh 2(\tau t - kx_0)}{\sinh^2 kH} \left[ 1 + \frac{3}{2} \frac{cos 2(\tau t - kx_0)}{\sinh^2 kH} \right] \]

where \( \eta_0 \) is the elevation of the agitated surface above the static level, and \( x \) and \( z \) are the wave profile equations.
Kozhevnikov [1] constructed potential wave profiles and determined the characteristics of wave motion by the method of electrohydrodynamic analogies.

An examination of Eqs. (5) and (6) shows that Nekrasov’s theory is linear, while Mich takes into account terms up to the second approximation with respect to the height of the wave. Herein lies the principal difference between them. Nekrasov assumed that his theory was valid only for very shallow waves \( h/\lambda \leq 1/38 \).

By way of example, Fig. 1 shows the experimental and theoretical wave profiles for \( h = 4.25 \text{ cm} \), \( \lambda = 50 \text{ cm} \), \( H = 9.5 \text{ cm} \), where the curves are numbered: 1) experimental; 2) Kozhevnikov; 3) Mich; 4) linear theory; 5) Nekrasov; 6) Stokes; and 7) trochoidal theory.

In this and other cases it was found that the best approximation to the experimental data is given by constructions based on Kozhevnikov’s graphs and computations based on Mich’s relations. Unfortunately, Kozhevnikov does not present formulas for the particle velocity and excess pressure corresponding to the wave profiles he obtained. Accordingly, for our purposes we shall use the equations of motion in the form proposed by Mich. These equations have already been used by D. D. Lappo [7], V. V. Khapskii, and G. G. Metelitsyna [8] for determining the wave pressure on certain types of hydroengineering structures.

The above-mentioned relations have the following form:

\[
\begin{align*}
\nu_x &= \frac{gh}{2} \frac{\sin k(H+z)}{\sin kH} \cos (at - kx) + \frac{3kh^3}{16} \frac{\sin 2k(H+z)}{\sin^3 kH} \cos 2(at - kx), \\
\nu_z &= -\frac{gh}{2} \frac{\sin k(H+z)}{\sin kH} \sin (at - kx) - \frac{3kh^3}{16} \frac{\sin 2k(H+z)}{\sin^3 kH} \sin 2(at - kx), \\
p &= \frac{pgh\sin k(H+z)}{2} \frac{\cos (at - kx)}{\sin kH} + \frac{3pgh^3 \sin k \sin kH}{16} \frac{\cos 2(at - kx)}{\sin^3 kH} - \\
&\quad - \frac{pgh^3 \sin^2 k \sin kH}{16} \frac{\cos 2(at - kx)}{\sin^3 kH} + \frac{pgh^3 \sin k \sin kH}{16} \frac{\cos 2(at - kx)}{\sin^3 kH}.
\end{align*}
\]

These equations hold true when for \( h/\lambda \leq 0.074 \) the ratio \( H/\lambda \geq 0.132 \), while for \( H/\lambda \leq 0.146 \) there are no restrictions on the steepness of the waves. Formula (9) above was obtained by Biesel [9], using the existing Mich solution, written in a somewhat different form.

Considering the second terms on the right sides of Eqs. (7), (8) and comparing the latter with expressions (1), (2), we see that the numerical values of the particle velocity, averaged over the period of the wave, as obtained from the formulas of the first and second approximations, are the same. On the other hand, for individual moments with respect to the phase of the wave motion the calculated values of the particle velocity may be considerably different, depending on whether formulas (1) and (2) or (7) and (8) are used for the purpose.

This is shown in Fig. 2, which for a complete period gives the results of computations (of the velocities in cm/sec and the pressure \( p \) in g/cm\(^2\)) based on formulas (1)-(8) and (7)-(9) for \( z = 0, h = 12.1 \text{ cm}, \lambda = 245 \text{ cm}, H = 36 \text{ cm} \). The individual graphs show: a) vertical projections of the particle velocity; b) horizontal projections of the particle velocity; c) orbital velocities; d) excess wave pressure; 1) second approximation; 2) linear theory; 3) corrections to second approximation. In all the graphs, and especially (c), it is clear that the second-approximation terms have a considerable influence in the individual phases of the period. It is interesting to consider the effect of the relative depth of the water on the terms of second approximation with respect to wave height. We shall do this in relation to the averaged values for the phase of passage of the wave crest, although, as noted above, this averaging leads to a certain drawing together of the results obtained from the theories of waves of small amplitude and finite height. We shall denote the averaged values with respect to the depth and the period of the wave by means of brackets, thus: \( <v_x> \), the auxiliary subscript indicating averaging over the period only. From Eqs. (7)-(9), averaging \( v_x, v_z \), and \( p \) with respect to the depth, for the first quarter period of the fundamental wave and the first quarter period of the superimposed wave, we get for the wave crest: