ELASTIC BENDING VIBRATIONS OF A ROD CARRYING ELECTRIC CURRENT

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ABSTRACT: In [1] a dispersion relation for the vibrations of an elastic rod of circular cross section with an electric current flowing over its surface was obtained, and a detailed study was made of the particular case of axially symmetric vibrations. The present paper is devoted to an examination of the longwave bending vibrations of elastic rods with an electric current flowing over their surface. These vibrations are of special interest since they have the lowest frequency and hence last the longest.

1. Investigation of bending vibrations of a circular rod on the basis of the general equations of the theory of elasticity. We shall consider a perfectly conducting solid rod of radius a with free ends and a constant current I flowing over its surface. Let the displacements of points on the rod be described by the vector

\[ \mathbf{u} = U(r) \exp \left( -\omega t + i\theta + k z \right) \]

The case of axially symmetric vibrations with m = 0 was thoroughly investigated in [1]. Here we shall consider the case of bending vibrations (m = 1). For an infinite rod the dispersion relation for such vibrations has the form

\[ \frac{d_{ij}}{d_{ij}} = 0 \quad (i, j = 1, 2, 3) \]

with the following elements of the determinant:

\[ d_{11} = 2\pi \left[ 1 + \psi_i (x) \right] + \frac{4}{1 + \nu} + \left( \nu^2 - \frac{1}{1 + \nu} \right) \psi_i (x), \]

\[ d_{22} = \frac{Y^2 - 1}{1 + \nu} \psi_i (Y) - \left( \nu^2 - \frac{1}{1 + \nu} \right) \psi_i (X), \]

\[ d_{33} = -1 \psi_i (X), \quad d_{23} = \psi_i (X) - \psi_i (Y), \]

\[ d_{33} = \psi_i (X) - \frac{Y^2 - 1}{2} \psi_i (Y), \quad d_{33} = h^2 - \frac{1}{1 + \nu}, \]

Here

\[ Y^2 = h^2 (\frac{\rho \sigma^2}{k^2 (h + 2\mu)} - 1) - \frac{1}{\nu} \left[ 2(2\mu + \nu) \right] \psi_i (x), \]

\[ Y^2 = h^2 \left( \frac{\rho \sigma^2}{k^2 \nu} - 1 \right) - \frac{1}{\nu} \left( 2(2\mu + \nu) - 1 \right), \]

\[ h^2 = \frac{H^2}{8\pi E} = \frac{I^2}{200\pi \mu^2 E}, \quad H_0 = \frac{2I}{10\pi}, \]

\[ \psi_i (\xi) = \frac{J_1 (\xi)}{\xi J_1 (\xi)}, \quad \psi_i (\xi) = \frac{K_1 (\xi)}{\xi K_1 (\xi)}, \]

where E is the modulus of elasticity, \( \lambda, \mu \) the Lamé coefficients, \( \nu \) Poisson’s ratio, \( \rho \) the density of the material, \( I \) the current in amps, \( H_0 \) the magnetic field at the surface of the conductor at \( r = a \), and \( J_i (\xi), K_i (\xi) \) are cylindrical functions.

Solving (1.1) for \( h^2 \), after certain transformations we get

\[ h^2 = \frac{\rho \sigma^2}{k^2 (h + 2\mu)} - 1 \quad (i, j = 1, 2, 3) \]

The elements of the determinant in (1.3) have the form

\[ a_{11} = \psi_i (X) - 2, \quad a_{11} = 1, \]

\[ a_{21} = \psi_i (X) - 1, \quad a_{21} = 2 \left[ \psi_i (Y) - 2 \right], \]

\[ a_{31} = -2 \left[ \psi_i (Y) - 2 \right] - 2 \left[ \psi_i (Y) - 2 \right], \]

\[ a_{32} = -2 \left[ \psi_i (Y) - 2 \right] - 2 \left[ \psi_i (Y) - 2 \right]. \]

In the case of long waves (\( ak << 1 \)) for the functions \( \psi_i (\xi) \) we have the following approximate expressions:

\[ \psi_i (\xi) \approx \frac{\pi}{\sqrt{2}} \frac{\left( \ln \frac{k a}{2} + C + \frac{1}{2} \right)}{k_0^2}, \]

Here \( k^2 = -\frac{\pi}{\sqrt{2}} \frac{\pi}{\sqrt{2}} \frac{\left( \ln \frac{k a}{2} + C + \frac{1}{2} \right)}{k_0^2}. \]

2. Approximate theory of longwave bending vibrations of an elastic rod carrying current. a) General relations. We shall consider a homogeneous cylindrical rod of arbitrary but constant cross section and infinite length. We shall assume that an electric current I flows over the surface of this rod. If the wavelength of the bending vibrations is much greater than the rod diameter, and the vibrations themselves are plane, then the equation of the vibrations may be written in the form [2]

\[ \frac{\partial^2 u}{\partial \xi^2} = - EJ \frac{\partial^2 u}{\partial z^2} + f_y + \ldots \]

Here A is the cross-sectional area of the rod; the wave is propagated in the direction of the rod axis z; \( w \) is the displacement in the direction of the y axis, perpendicular to the axis of the rod, \( J \) is the moment of inertia of the rod cross section, and \( f_y \) the external force acting on unit length of the rod in the direction of the displacement \( w \). In the case of longwave vibrations the discarded terms in (2.1) have a higher order of smallness, in the case of a current-carrying rod the force \( f_y \) owes its manifestation to the magnetic field, if we take two sections perpendicular to the axis of the undeformed rod and a distance \( d \) apart, then

\[ \frac{1}{l^2} = - \frac{\partial^2 u}{\partial z^2} = - w^2, \quad H = H_0 + H_1, \]
where $H_2$ is the perturbation of the magnetic field. Then expression (2.2) for the force $f$ can be linearized with respect to perturbations of the magnetic field [3]. For $f_y$ we have

$$f_y = \frac{w}{8\pi} \int \frac{H_x \phi_y - \frac{1}{\alpha^2} H_y \phi_x}{(\alpha^2)} \, ds \equiv f_1 + f_2, \quad (2.4)$$

Here we have taken into account the fact that in the absence of vibrations $H = H_0$, $f = 0$, while $H_2$ denotes the value of the magnetic field perturbation at the surface $S$. From (2.4) it is clear that the calculation of $f_2$ presupposes determination of the field perturbation $H_1$.

In computing the perturbed field $H$ it is natural to use the scalar potential $\Phi; H = \nabla \Phi$, where $\Phi$ satisfies the Laplace equation $\Delta \Phi = 0$. On the surface of the conductor, in virtue of the assumption made above, that the entire current flows over the surface, the field must satisfy the boundary condition

$$H \cdot n = 0, \quad (2.5)$$

Let the deformed axis of the rod be described by the equation $y = w(x, t)$. We shall go over to a new coordinate system $X, Y, Z$ linked with the old one by the relations

$$X = x, \quad Y = y - w, \quad Z = z.$$

In the new system the equation $\Delta \Phi = 0$ takes the form

$$\Delta x, X_0 = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial z} = 0. \quad (2.6)$$

We shall solve this equation by the method of perturbations. We set $\phi = \phi_1 + \phi_2$. The potential of the unperturbed field satisfies the equation

$$\Delta x, \phi_0 = 0. \quad (2.7)$$

Assuming that the bending of the conductor is small, we shall neglect terms of the second order of smallness in $w, \phi_1$ and their derivatives.

Then for $\phi_1$ from (2.6) we get the equation

$$\Delta x, \phi_0 = 0. \quad (2.10)$$

The solution $\phi_0$ of Eq. (2.10) can be represented as the sum of the solutions $\psi$ of the homogeneous equation and the particular solution $\psi'$ of the inhomogeneous equation

$$\psi = \psi + \psi'. \quad (2.9)$$

In connection with the decomposition (2.9) of the field perturbation into two parts, it is likewise natural to divide the force $f_2$ in (2.4) into two components,

$$f_2 = f' + f'.$$  

(2.10)

The inhomogeneous equation has the particular solution

$$\psi' = w \frac{\partial \phi_0}{\partial y}. \quad (2.11)$$

The solution of the homogeneous equation

$$\left(\Delta x, \phi_0 + \frac{\partial^2 \phi_0}{\partial y^2}\right) \psi = 0. \quad (2.12)$$

is uniquely determined by boundary condition (2.5). Setting

$$\psi = w \Psi (X, Y), \quad w = w_0 \exp ikZ, \quad (w_0 = w_0(t)), \quad (2.13)$$

we can rewrite (2.12) in the form

$$\left(\Delta x, \Psi - k^2 \right) \Psi = 0. \quad (2.14)$$

Since in deriving (2.1) it was assumed that $k \to 0$, for our purposes Eq. (2.14) can be solved approximately, with account only for the terms containing lower powers of $k^2$. However, in simple cases it is more convenient to start from exact solutions of (2.14) and perform the expansion in powers of $k^2$ in the final formulas. This is the method used below.

b) Bending vibrations of a circular rod. Using the cylindrical system of coordinates $R, \phi, Z(X = R \cos \phi, Y = R \sin \phi)$, we can write the scalar potential of the unperturbed magnetic field of a circular rod $\psi_0$, the solution of Eq. (2.7), as

$$\psi_0 = \Psi \phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Phi \Fi