We consider the oblique incidence of a sinusoidal electromagnetic wave (with the electric vector in the plane of incidence) on a parabolic layer of a plane-layered isotropic plasma. The reflection and transmission factors are obtained, and also the electromagnetic-plasma wave conversion factor.

1. Consider the oblique incidence of a sinusoidal wave, with the electric vector in the plane of incidence, on a parabolic layer of a plane-layered isotropic plasma. The electric and magnetic waves in the plasma, in this case, are described by the following equations [1]:

\[ E_y = -\frac{i}{\varepsilon_0 c} \frac{\partial H_x}{\partial z}, \quad E_z = -\frac{i}{\varepsilon_0 c} \frac{\partial H_y}{\partial y} \]  \hspace{1cm} (1.1)

\[ \frac{\partial G}{\partial z} - \frac{1}{\varepsilon'} \frac{\partial^2 G}{\partial z^2} + \Omega^2 (\varepsilon' - \varepsilon_0^2) G = 0 \quad (\Omega = \frac{\omega}{c}, \quad \varepsilon_0 = 2\pi) \]  \hspace{1cm} (1.2)

\[ \varepsilon' = 1 - \frac{\varepsilon_0^2}{\varepsilon} \left( 1 - \frac{1}{\varepsilon_0^2} \right) - is, \quad \varepsilon_0 = \sin \theta_0, \quad i = \sqrt{-1} \]  \hspace{1cm} (1.3)

Here \( f \) is the frequency, \( c \) is the speed of light, \( \theta_0 \) is the angle of incidence of the wave on the layer, \( \varepsilon' \) is the complex dielectric constant, \( f_* \) is the critical frequency, and \( z_m \) is the half-thickness of the layer. We assume that the absorption \( s \) is constant along the layer. The case of a linear layer has been considered in detail in [2]. The case of a second-order zero (\( s = 0, f = f_* \), \( \varepsilon' = \varepsilon = z^2/z_m^2 \)) was partly considered in [3]; however, the field was considered mainly in the region \( \varepsilon' = 0 \). The reflection and transmission factors were not calculated explicitly and were not analyzed. Since the wave equation can be solved accurately for \( \varepsilon' = z^2/z_m^2 \), it is of special interest to obtain these factors. We therefore consider them first.

2. Consider the particular case

\[ \varepsilon' = \varepsilon = z^2/z_m^2, \quad s = 0, \quad f = f_* \]

With these assumptions Eq. (1.2) becomes

\[ \frac{\partial G}{\partial z} + \frac{Q}{2} \frac{\partial G}{\partial z} + (a + bx^2) G = 0 \]  \hspace{1cm} (2.1)

In this case the arbitrary parameters \( Q, a, \) and \( b \) take the values

\[ Q = -2, \quad a = -\Omega^2 \omega_0^2, \quad b = \omega_0^2/c^2 z_m^2, \quad \omega_0 = 2\pi f_* \]

The \( G(1) \) and \( G(2) \) solutions of this equation can be expressed in terms of Whittaker functions [4]

\[ G^{(1)}(\tau) = e^{\pm i\pi} (i)^{\kappa_2} \text{W}_{\kappa_2, \rho} (\tau \varepsilon^{-1}) \sim \tau^{\kappa_2} e^{\pm i\pi \rho} (1 + O (\tau^{-1})) \quad (|\tau| \to \infty) \]

\[ G^{(2)}(\tau) = (i)^{-\kappa_2} \text{W}_{\kappa_2, \rho} (i \tau \varepsilon^{-1}) \sim \tau^{\kappa_2} e^{\pm i\pi \rho} (1 + O (\tau^{-1})) \quad (|\tau| \to \infty) \]

\[ \tau = z^2/b, \quad k = a/c, \quad j = \frac{1}{4} (r_2 - r_1), \quad \mu = \frac{1}{4} (p_1 - p_2), \quad r_{1,2} = \frac{1}{8} (1 \pm 4s b^{-1/2}) \]  \hspace{1cm} (2.2)
Here $\rho_1$ and $\rho_2$ represent the behavior of the solutions in the neighborhood of the ordinary singular point of the equation $z = 0$, and can be obtained from the defining equation

$$\rho (\rho - 1) + \rho Q = 0 \quad (2.3)$$

If the incident wave propagates from the side $z < 0$, then $G^{(2)}$ describes a transmitted wave for $z > 0$. Correspondingly, for $z < 0$, $G^{(2)}$ describes a reflected wave and $G^{(1)}$ describes an incident wave. Starting from Eqs. 9.120, 9.231 (2), and 9.232 of [4], we can obtain

$$W_{\eta, \mu} (z e^{\pm \pi i}) = \frac{2\pi i e^{-i\pi \eta} W_{\eta, \mu} (z e^{i\pi \eta})}{\Gamma(\frac{1}{2} + \mu - \eta) \Gamma(\frac{1}{2} - \mu - \eta)} - (q_\pm 2\cos 2\mu + e^{i\pi \eta}) W_{\eta, \mu} (z)$$

$$q_+ = 1, \quad q_- = 0 \quad (\Gamma \text{ is a gamma function})$$

(A formula for $W_{\eta, \mu} (z e^{i\pi \eta})$ is given in [3], but due to a misprint $e^{i\pi \eta}$ is written for the second term.)

From (2.2) and (2.4), the following relationship between the solutions $G^{(1)}$ and $G^{(2)}$ is obtained on the positive and negative semiaxes $z$:  

$$G^{(2)} (i e^{-i\pi \eta}) = -\frac{2\pi i e^{-i\pi \eta} G^{(1)} (i \eta)}{\Gamma(\frac{1}{2} + \mu - \eta) \Gamma(\frac{1}{2} - \mu - \eta)} - i (q_\pm 2\cos 2\mu + e^{-i\pi \eta}) G^{(1)} (i \eta)$$

Equation (2.5) in particular, gives the relationship between the asymptotic solutions of Eq. (2.1) on the positive and negative semiaxes $z$, as $|\tau| \to \infty$. As can be seen from (2.5), the nature of this relationship depends on the direction in which point $z$ is circumvented (over the upper ($q_+$) or lower ($q_-$) half-planes of the complex $z$-plane). This means that the solutions in $z = 0$, generally speaking, have a branch point. Hence, in general, the problem arises as to the correct choice of the direction in which one circumvents the point (see section (4.5)). The following expressions are obtained from (2.5) for the amplitude reflection factor $R$ (the ratio of the reflected wave amplitude to the incident wave amplitude) and for the transmission factor $D$:

$$D = (2\pi)^{-1} e^{i\pi (\tau - \epsilon)} \Gamma(\frac{1}{2} + \mu - \eta) \Gamma(\frac{1}{2} - \mu - \eta)$$

$$R = e^{i\pi \tau} (q_\pm 2\cos 2\mu + e^{-i\pi \eta}) D$$

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In the case considered $Q = -2$, $\mu = 3/4$, $\cos 2\pi \mu = 0$, i.e., the directions of rotation are equivalent. Substituting the actual expressions for $Q$, $a$, and $b$ in (2.6), and also using the properties of gamma functions [see [4], Eq. 8.331, 8.335 (1), and 8.332 (2)], we can obtain the following expressions for the reflection factor $|R|^2$ (the ratio of the reflected wave intensity to the incident wave intensity) and the transmission factor $|D|^2$:

$$|R|^2 = \frac{e^h}{1 + e^h}, \quad \frac{|D|^2 = \frac{1}{1 + e^h}}{h = \frac{2\pi^2 \kappa_m}{\kappa} \xi_0^2}$$

Equation (2.5) in particular, gives the relationship between the asymptotic solutions of Eq. (2.1) on the positive and negative semiaxes $z$, as $|\tau| \to \infty$. As can be seen from (2.5), the nature of this relationship depends on the direction in which point $z$ is circumvented (over the upper ($q_+$) or lower ($q_-$) half-planes of the complex $z$-plane). This means that the solutions in $z = 0$, generally speaking, have a branch point. Hence, in general, the problem arises as to the correct choice of the direction in which one circumvents the point (see section (4.5)). The following expressions are obtained from (2.5) for the amplitude reflection factor $R$ (the ratio of the reflected wave amplitude to the incident wave amplitude) and for the transmission factor $D$:

$$D = (2\pi)^{-1} e^{i\pi (\tau - \epsilon)} \Gamma(\frac{1}{2} + \mu - \eta) \Gamma(\frac{1}{2} - \mu - \eta)$$

$$R = e^{i\pi \tau} (q_\pm 2\cos 2\mu + e^{-i\pi \eta}) D$$

$$|R|^2 = \frac{e^h}{1 + e^h}, \quad \frac{|D|^2 = \frac{1}{1 + e^h}}{h = \frac{2\pi^2 \kappa_m}{\kappa} \xi_0^2}$$

Hence we see that for $s = 0$ and $f = f_0$ there is no absorption in the region of the poles ($|R|^2 + |D|^2 = 1$), and the presence of an infinite electric vector amplitude ($|E_\xi| \sim 1/z^2, |E_\phi| \sim 1/z$ as $z \to 0$) must be regarded as the result of a transition process [1]. Putting $G = w e^{i\phi} Eq. (2.1)$ takes the form

$$\frac{\partial^2 w}{\partial \tau^2} + q^2 w = 0, \quad q^2 = \frac{2}{z^2} + \frac{\omega_0^2}{c^2} (\frac{z^2}{\xi_0^2} - \xi_0^2)$$

Figure 1 shows $q^2$ as a function of $z$. In the above calculation we have ignored reflection from the jump $\partial w/\partial z$ at the points $z = \pm z_m$. This is valid when $(z_m - z_0) / \lambda \to 1$ (Fig. 1) and the transition to $\epsilon = 1$ can be smoothed out in the same way. This condition formally limits the applicability of (2.7) to large $\theta_0$. However, in practice, total reflection from the layer begins at values of $\theta_0$ for which (2.7) holds. If, as an example, for $z_m / \lambda = 10$ it is required that $(z_m - z_0) / z_m \approx 1/2$, the condition $\theta_0 \approx 30^\circ$ ($h \approx 48.9$) must be satisfied. In fact, total reflection occurs for $\theta_0 < 30^\circ$.

3. Let us consider the relationship between the asymptotic solutions in the general case. For a parabolic layer Eq. (1.2) has the form

$$\frac{d^2 G}{dz^2} - \left(\frac{1}{z + s_1} + \frac{1}{z - s_1}\right) \frac{dG}{dz} + \Omega^2 (z^2 - s_1^2 - \xi_0^2) G = 0$$