Spatial dynamics of time periodic solutions for the Ginzburg-Landau equation

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1. Introduction

Modulation equations have been used to study the behavior of patterns at near critical conditions. Probably the most well known is the complex Ginzburg-Landau (CGL) equation,

\[ A_t = (\tilde{\alpha} A_{xx} + (R - R_c)A + \tilde{\beta}|A|^2A, \]

(1.1)

with \( \tilde{\alpha}, \tilde{\beta} \in \mathbb{C}, \text{Re}(\tilde{\alpha}) > 0, A = A(x, t) \) is complex valued, and \((x, t) \in \mathbb{R} \times \mathbb{R}^+\). The real number \( R \) is a control parameter for the underlying physical system, and \( R_c \in \mathbb{R} \) is a value at which a basic laminar solution bifurcates (either sub- or super-critically) to a periodic pattern. \( A \) describes the amplitude of the envelope modulating the bifurcating pattern.

Assuming that the bifurcation is subcritical, which we do in this paper, \( R < R_c \) and \( \text{Re} \tilde{\beta} > 0 \). Upon rescaling, the CGL then becomes

\[ A_t = (1 + i\mu)A_{xx} + (R - R_c)A + (1 + iq)|A|^2A, \]

where \( \mu, q \in \mathbb{R} \). Without loss of generality, it can be assumed that \( R - R_c = -1 \), which yields for (1.1)

\[ A_t = (1 + i\mu)A_{xx} - A + (1 + iq)|A|^2A. \]

(1.2)

In this paper we are interested in studying the spatial structure of time periodic solutions. Under the ansatz

\[ A(x, t) = a(x) e^{i\sigma t}, \]

where \( a: \mathbb{R} \to \mathbb{C} \) and \( \sigma \in \mathbb{R} \), this means that we are interested in finding

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solutions to
\[(1 + i\mu)a'' - (1 + i\sigma)a + (1 + iq)|a|^2a = 0 \quad \left(\frac{d}{dx}\right). \tag{1.3}\]

In studying the dynamics associated with (1.3), we will be particularly interested in finding solutions \(a(x)\) which are either periodic, quasiperiodic, or pulses \(|a(x)| \to 0\ as \ |x| \to \infty\). For simplicity only, we will set \(\mu = 0\) in (1.3). This is not necessary, however, as the behavior we describe in this paper occurs for all parameter values \(\mu, \sigma,\) and \(q\), with \(\sigma \neq \mu\), for which the conclusion of Proposition 1.1 holds. Under the restriction that \(\mu = 0\), we will then study the ODE
\[a'' - (1 + i\sigma)a + (1 + iq)|a|^2a = 0. \tag{1.4}\]

In this paper we will not discuss the dynamics of (1.4) for all possible values of \(\sigma\) and \(q\). Instead, we will be interested in a domain in parameter space in which \(\sigma\) and \(q\) are close to values in which a pulse solution exists.

**Proposition 1.1** (Hocking and Stewartson [7]). When
\[q = \frac{\sigma}{\frac{1}{3}\sqrt{1 + \sigma^2}},\]
there exist pulse solutions for (1.4).

**Remark 1.2.** More generally, Hocking and Stewartson show that \(q = q(\sigma, \mu)\).

**Remark 1.3.** We will let \(\sigma^* = \sigma^*(q)\) denote the value for which the pulses exist. Note that \(\sigma^*\) is a monotonically increasing function of \(q\), and that \(q \neq 0\) implies that \(\sigma^* \neq 0\).

**Remark 1.4.** Throughout this paper it will be assumed that \(|\sigma - \sigma^*| \leq 1\).

It is important to note the symmetries associated with (1.4). First, observe that if \(a(x)\) is a solution to (1.4), then so is \(a(x) e^{i\theta}\), where \(\theta \in [0, 2\pi)\). This invariance allows a reduction in dimension for the resulting ODE. Setting \(a(x) = b(x) + ic(x)\) in (1.4), the second order complex ODE (1.4) becomes the first order system
\[b' = d\]
\[c' = e\]
\[d' = b - \sigma c - (b - qc)(b^2 + c^2)\]
\[e' = \sigma b + c - (qb + c)(b^2 + c^2). \tag{1.5}\]