A connection is discussed between the group SU(2) and supersymmetry for a series of quantum mechanical problems. It is pointed out that the impossibility of factorizing Hamiltonians obtained based on representations of the group SU(2) indicates that the supersymmetry of the system is broken.

A very important problem in quantum mechanics is the study of eigenfunctions and eigenvalue spectra for various Hamiltonians which can be used as a basis for constructing models of various physical systems. A wide spectrum of exactly solvable quantum mechanical problems has been considered in detail in [1] and [2], where the Schrödinger, Klein-Gordon, and Dirac equations have been solved for various configurations of external electromagnetic fields. In addition, we have recently obtained certain exact solutions of the Schrödinger equation for double-well potentials which appear in a series of problems in solid-state physics, field theory, and other branches of physics. The representation theory for the groups SU(2) or SO(3) was essential in obtaining these solutions. It is also known that the factorization method [3-5] and the ideas of supersymmetry allow us to broaden substantially our understanding of properties of systems described by appropriate types of equations.

Having in mind the above fact, in this brief note we consider a solution of the anharmonic oscillator problem, and we study properties of solutions for a series of problems for which supersymmetry is possible.

In previous work we have constructed, using a representation of the group SU(2), a series of Hamiltonians describing a multiparameter anharmonic oscillator. The analytic expression for the Hamiltonian with the number \( n = 2s \) is

\[
H_n = \alpha \frac{\partial^2}{\partial x^2} + \beta |x|^\gamma \left( x^2 - \frac{1}{2} \beta \right) x - \left( (1 + 2s) \gamma + \nu \right) |x| + (\beta 2)^s - (1 + 2s) x \right).
\]

For every Hamiltonian, i.e., for \( s = 0, 1/2, 1, 3/2, \ldots \), and for certain values of the energy \( E_k \), exact solutions of the Schrödinger equation (SE)

\[
H_n \Psi_n = E \Psi_n
\]

have been found. As has been shown previously, determining the energy eigenvalues simply reduces to solving an algebraic problem. For wave functions satisfying Eq. (2) we have found the following representation:

\[
\Psi_n (z) = \Psi_0 (z) \sum_{\kappa=1}^{n} \frac{C_{\kappa}}{(s-\kappa)! (s+\kappa)!} z^{s+\kappa}
\]

where \( \Psi_0 (z) = \exp \left( \frac{1}{2} (\beta |z|^\gamma - az^2 - 1/3 \gamma |z|^3) \right) = \exp \left( -\mathcal{W}(z) \right) \) is a solution of Eq. (2) for \( s = 0 \), \( \alpha \) and \( \beta \) are arbitrary parameters, \( \gamma > 0 \), and the coefficients \( C_\kappa \) are related to each other by a recurrence relation.
\[
\frac{\gamma (s + \kappa + 1) C_{k-1}}{V (s - \kappa + 1)! (s + \kappa - 1)!} - \frac{(2\kappa - E) C_{k}}{V (s - \kappa)! (s + \kappa)!} + \frac{\gamma (s - \kappa + 1) C_{k+1}}{V (s - \kappa - 1)! (s + \kappa + 1)!} + \frac{(s - \kappa + 1)(s - \kappa + 2) C_{k+2}}{V (s - \kappa - 2)! (s + \kappa + 2)!} = 0.
\] (4)

It is clear from the representation (3) that the wave functions are normalizable for arbitrary \(s\) (\(s\) can be integer or half-integer). This last property is extremely important in the study of eigenfunctions and eigenvalues of Hamiltonians of the form (1). Evidently, the normalizability property disappears if the sign in front of the parameter \(\gamma > 0\) is changed. Before proceeding to discuss further the properties of Hamiltonians defined by Eq. (1), we note that for \(s = 0\), the potential \(U(0, z)\) and the wave function (3) in the ground state \(\Psi_0 = C_0 \exp (-\hat{W}(z))\) are simply related to each other:

\[
U(0, z) = - (\hat{W}_0 + \hat{W}_1) z^2. \tag{5}
\]

The latter property of the potential and the wave function for \(s = 0\) indicates a connection between the expression (1), obtained by us, and a supersymmetric quantum mechanical Hamiltonian constructed by Witten [6]. This connection arises because for \(s = 0\) the Hamiltonian (1) can be factorized. (Necessary and sufficient conditions for factorization of Hamiltonians are formulated in Theorems 1-5 and paragraph 3 of the paper by Infeld and Hull [4].)

In order to factorize the Hamiltonian we choose the first-order differential operators \(L^+\) and \(L^-\) in such a way that they can be used to express Hamiltonian (1) for \(s = 0\). We note that for any other value of the index \(s\) the potential \(U(s, z)\) does not satisfy the factorization conditions given in [4]. It is clear that the following operators can be taken as the linear operators \(L^+\) and \(L^-\):

\[
L^+ = \partial_z + \hat{W}, \quad L^- = (L^+)^*, \tag{6}
\]

where \(*\) denotes conjugation. It is easy to see that the commutation relations for the operators \(L^-\) and \(L^+\) have the form \([L^-, L^+] = 2z^2 \bar{W}\). Using the chosen operators, we can write the Hamiltonian \(H_0\) in a factorized form

\[
H_0 = L^- L^+. \tag{7}
\]

We note that representation of the Hamiltonian in the form (6), through noncommuting operators, takes place in supersymmetric theory (see, e.g., [7]). In this case the simplest supersymmetric matrix Hamiltonian constructed from noncommuting operators \(L^+\) are \(L^-\) can be represented in the following way:

\[
\hat{H} = \begin{pmatrix} L^- & 0 \\ 0 & L^+ \end{pmatrix} = \begin{pmatrix} H_0 & 0 \\ 0 & H_{-2} \end{pmatrix} = \begin{pmatrix} \partial_z^2 + \partial_z \hat{W} - (\partial_z \hat{W})^2 & 0 \\ 0 & \partial_z^2 - \partial_z \hat{W} + (\partial_z \hat{W})^2 \end{pmatrix}. \tag{8}
\]

The operators \(\hat{L}^+ = L^- E_+\), \(\hat{L}^- = L^+ E_+\), (where, as previously, \(E_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\), \(E_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\)) satisfy the following relations: \(\hat{L}^+ \hat{L}^- = 0\), \(\hat{H} = (\hat{L}^+, \hat{L}^-)\), \([\hat{H}, \hat{L}^+] = 0\), \([\hat{H}, \hat{L}^-] = 0\), \([\hat{L}^+, \hat{L}^-] = 0\). i.e., the Hamiltonian \(\hat{H}\) really corresponds to the Hamiltonian of supersymmetric quantum mechanics, and the quantities \(L^\pm\) to supercharges.

We further note that the Hamiltonian \(H_0 = L^+ L^-\) and \(H_{-2} = L^- L^+\) are related to each other through the Darboux transformation [3]:

\[
H_0 = H_{-2} + [L^-, L^+] = \partial_z^2 - \partial_z \hat{W} - (\partial_z \hat{W})^2 - 2\partial_z \hat{W} = \partial_z^2 + \partial_z \hat{W} - (\partial_z \hat{W})^2. \tag{9}
\]

The relation for wave functions is also known

\[
L^- \Psi_0(E_0) = 0; \quad \Psi_0(z, E_i) = \frac{1}{E_i} L^* \Psi_{-2}(z, E_i). \tag{10}
\]

Our analysis demonstrates that for the value \(s = 0\) the Darboux transformation allows us to obtain the Hamiltonian \(H_{-2}\), which can thus be consolidated with the initial supersymmetric matrix Hamiltonian. The above-noted property of the system representing the anharmonic oscillator attests to its rich internal structure and also to the complexity of physical systems which are described by this model. It is interesting that the spectra of Hamiltonians \(H_0\) and \(H_{-2}\) are similar to each other; they differ only by the absence of the level \(E_0 = 0\) from the spectrum of \(H_{-2}\). This property of the spectra of operators \(H_0\) and \(H_{-2}\) can turn out to