GROUP CLASSIFICATION AND INVARIANT SOLUTIONS FOR THE EQUATIONS OF FLOW AND HEAT TRANSFER OF A VISCOPLASTIC MEDIUM

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Exact solutions without major restrictions on the properties of the material are needed in research on the flow (especially viscosity) of metals at high temperatures under nonisothermal conditions. Often the shear resistance is governed mainly by the temperature and the deformation rate. Here we examined the group properties of the equations of flow and heat transfer of a medium whose shear resistance is a function of temperature and rate of shear deformation. The properties specific to metals are not used, so the results are applicable to a variety of media.

§1. Here we consider three types of flow accompanied by heat transfer for a medium filling a finite or infinite region $x > x_0 (x_0 = 0)$.

1. Planar rectangular flow without a pressure gradient caused by motion of a boundary in a direction perpendicular to the $x$ axis.

2. Rectilinear flow with axial symmetry without a pressure gradient caused by translational motion of a circular cylinder of radius $x_0$ in the direction of the generator.

3. Flow caused by the rotation of a circular cylinder of radius $x_0$ about its axis.

It is assumed that the shear stress is a function of the temperature and deformation rate. These simple types of flow allow one to obtain exact solutions without further assumptions about the properties of the medium.

The equation of motion and the equation of heat flow are

$$
\rho \frac{\partial v}{\partial t} - \frac{\partial \Phi (e, T)}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial \Phi (e, T)}{\partial T} \frac{\partial T}{\partial x} + \frac{\partial \Phi (e, T)}{\partial e} \frac{\partial e}{\partial x} + \frac{\Phi (e, T)}{x} \frac{\partial v}{\partial x} = 0,
$$

$$
\rho c \frac{\partial T}{\partial t} - \frac{\partial \lambda (T) \frac{\partial T}{\partial x}}{\partial x} - \frac{\partial \lambda (T) \frac{\partial T}{\partial x}}{\partial t} - \rho \frac{\partial (L (T))}{\partial t} = 0.
$$

The above three types of flow correspond to the following combinations of $\delta_1$ and $\delta_2$: 1) $\delta_1 = \delta_2 = 0$; 2) $\delta_1 = 1, \delta_2 = 0$; 3) $\delta_1 = \delta_2 = 1$; $v$ is the corresponding velocity component, $T$ is temperature, $\rho$ is density, and $c$ is specific heat. The function $\Phi (e, T)$ gives the shear stress $\tau$ as a function of $T$ and the deformation rate in shear:

$$
\tau = F (e, T) \left( \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \right),
\varepsilon = \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y},
\Phi = Fe;
$$

this, the thermal conductivity $\lambda (T)$, and the function $L(T)$ (representing heat released by phase transformations) allow a certain range of choice in their forms. It is assumed that the heat produced as a result of the viscosity may be neglected.

We define $T^*, f(T^*)$, $\Phi (e, T^*)$ as follows:

$$
T^* = \rho \left[ c T - L (T) \right],
$$

$$
f (T^*) = \lambda / (e - L'),
\Phi^* = \Phi / \rho .
$$

System (1.1) becomes as follows in the new variables (the superscript is omitted):

$$
\left\{
\begin{array}{ll}
\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial T} \frac{\partial T}{\partial x} + \frac{\partial \Phi}{\partial e} \frac{\partial e}{\partial x} & = 0,
\\
\frac{\partial T}{\partial t} - \frac{\partial \lambda (T) \frac{\partial T}{\partial x}}{\partial x} - \frac{\partial \lambda (T) \frac{\partial T}{\partial x}}{\partial t} - \frac{\partial L (T)}{\partial t} & = 0,
\\
\frac{\partial e}{\partial x} - \frac{\partial v}{\partial x} e & = 0.
\end{array}
\right.
$$

This is the system that will be examined.

§2. Consider the group properties of system $S$ in accordance with the general methods of [1-3], which have [4-6] been applied to various physical problems. We consider the invariance of $S$ relative to the operator

$$
X = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial \eta}
$$

of the sought group $G$, the conditions for this being complied with on a manifold specified by $S$; this gives us a system of equations of the Lie algebra of the basic group, which we write out as follows after simplification:

$$
\eta_0 \frac{\partial \eta}{\partial t} - 2 \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t} = 0,
\frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial t} = 0,
$$

$$
\frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial t} = \frac{\partial \eta}{\partial t} = 0,
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$$
Here $R$ is the Reynolds number, $A$ is the Alfven number, $R_m$ is the magnetic Reynolds number, $P$ is the Prandtl number, $U$ is the velocity of the unperturbed flow, $B$ is the unperturbed magnetic field, $T$ is the unperturbed temperature, while $\sigma$, $\kappa$ are the electrical and thermal conductivity and viscosity in the unperturbed flow. The primes denote differentiation with respect to $y$.

As usual the solution of the system is written in the form

$$\psi = \psi(y) \exp ik(x - ct), \quad (2.5)$$

where $k$ is the dimensionless wave number and $kc$ is the dimensionless frequency of the oscillations. Equations (2.2)-(2.4) must be solved for the following obvious conditions:

$$\psi(\pm 1) = \psi'(\pm 1) = 0, \quad \Theta(\pm 1) = 0. \quad (2.6)$$

The boundary conditions for the magnetic field in the case of nonconducting walls have the form

$$(\psi'/\psi)_{\pm 1} = \mp k. \quad (2.7)$$

If system (2.2)-(2.4) is not separable, then hydrodynamic, electrodynamic, and thermal effects exert a simultaneous influence on the stability.

3. The Overheat Instability. We shall first of all consider the case $S \ll R_m$, where $S = M^2 / R$ is the hydromagnetic interaction parameter. Clearly in this case field perturbations caused by the motion of the medium may predominate over velocity perturbations caused by the field. In the limit for $A \to 0$ for $\gamma = 0$ we may imagine a situation when the velocity perturbations also tend to zero, and the terms containing $\psi$ in Eqs. (2.3), (2.4) may be neglected. If we make the further assumption that $R_m \ll 1$, then we have from (2.3)

$$\Delta \psi = \alpha B_s \Theta. \quad (3.1)$$

Using (2.4), (2.5), and (3.1) and neglecting for simplicity the contribution of viscous dissipation and the fact that $\kappa$ is not constant, we obtain, after making formal transformations,

$$\Theta'' + (E - V) \Theta = 0, \quad E = -k^2 + ikc R P, \quad V = -\alpha II \frac{\gamma'}{\gamma} + ikc R P \quad (\Pi = \frac{\gamma'}{\gamma}). \quad (3.2)$$

The problem thus becomes one of finding the eigenvalues of the Schrodinger equation with a complex potential $V$. If the initial steady state is symmetric with respect to $y$, then it is not hard to see that $Re V$ is a "potential well," and $Im V$ has the form of a hump. The potential may be expanded in a series to give the Schrodinger equation for a harmonic oscillator in the region of the axis of the channel. Having thus ascertained that finite solutions exist [11], we may employ simple approximate methods in order to investigate (3.2). For example in the quasi-classical approximation we replace $d/dy$ by $ik \gamma$ and obtain the stability criterion immediately (in dimensional form):

$$\sigma \delta^{4/3} > \frac{d \ln z}{d \ln T} \frac{R^2}{c}. \quad (3.3)$$

Formula (3.3) was obtained previously for the general case in paper [4], but the question of the existence of finite solutions was not considered. The presence of the factor $\kappa$ in inequality (3.3) prompts us to call the instability an overheat instability [5, 7]. For simplicity we shall restrict ourselves to considering the case $S \ll R_m \ll 1$ in the quasi-classical approximation. A similar analysis may be carried out without this last restriction.

4. Hydrodynamic Instability. We shall now consider the other limiting case in which the instability is caused by the purely hydrodynamic mechanism of the untwisting of the velocity gradient vortex. It is well known that the onset of hydrodynamic instability occurs for fairly large Reynolds numbers $R$. We may therefore neglect the small terms in the right-hand side of (2.2), retaining, however, the old derivative. Further we shall confine ourselves to the case $R_m \ll 1$, where we can neglect terms containing $B_x$ compared with $B_y$. From Eq. (2.3) we have

$$\sigma \delta^{4/3} > \frac{d \ln z}{d \ln T} \frac{R^2}{c}. \quad (3.4)$$

If the hydromagnetic interaction parameter $S \ll 1$, i.e., the Hartmann number is not very large, then we may eliminate $\psi$ from (2.2) using (4.1) and, neglecting small terms, finally arrive at a problem which is one of finding the eigenvalues for an Orr-Sommelherd type equation

$$\left(U - c\right) \left(\psi'' - k^2 \psi\right) - U^2 \psi = \frac{1}{ikc} \eta \psi'' \Psi \quad (4.2)$$

with boundary conditions (2.6). Thus for $R_m \ll 1$, $S \ll 1$, $\alpha S < 1$ the magnetic field and nonisothermal nature of the flow exert an indirect influence on the stability of the motion, altering the velocity profile and introducing a viscosity profile into Eq. (4.2). In order to solve the problem we use the familiar Heisenberg-Lin method [9]. We shall, as usual, confine ourselves to treating even perturbations over the channel half-