CLASSIFICATION OF THE INVARIANT SOLUTIONS TO THE EQUATIONS FOR THE TWO-DIMENSIONAL TRANSIENT-STATE FLOW OF A GAS

N. Kh. Ibragimov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 7, No. 4, pp. 19-22, 1966

Here one considers all invariant solutions to the system of equations for two-dimensional gas dynamics:

\[ \frac{\partial v}{\partial t} + (v \cdot \nabla) v + \frac{1}{\rho} \text{grad} p = 0, \]

\[ \frac{\partial \rho}{\partial t} + (v \cdot \text{grad} \rho) + \rho \text{div} v = 0, \]

\[ \frac{\partial \rho}{\partial t} + (v \cdot \text{grad} \rho) + \Delta \text{div} v = 0, \]

\[ \left( A = A(\rho, \rho) \equiv -\rho \frac{\partial S}{\partial \rho} \right). \]

Here \( p \) is pressure, \( \rho \) is density, \( S \) is entropy, and \( v = v(x,y) \) is the velocity vector, whose components are \( u \) and \( v \); it is assumed that \( \partial S/\partial \rho \neq 0 \). Two cases will be considered.

Case A: \( A(\rho, \rho) \) an arbitrary function.

Case B: \( A = \gamma p \), a polytropic gas with \( \gamma = \text{constant} \).

The principal group of transformations allowed by (1) has been given [1], and for case A the basis of the corresponding Lie algebra consists of the operators

\[ X_1 = \frac{\partial}{\partial t}, \quad X_4 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_5 = t \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \rho}, \]

\[ X_2 = \frac{\partial}{\partial x}, \quad X_6 = t \frac{\partial}{\partial x} + x \frac{\partial}{\partial \rho} + y \frac{\partial}{\partial \rho}, \]

\[ X_3 = \frac{\partial}{\partial y}, \quad X_7 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial \rho} + v \frac{\partial}{\partial u} - u \frac{\partial}{\partial \rho}. \]

while in case B we add to these the operators

\[ X_8 = t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} + v \frac{\partial}{\partial \rho} + 2p \frac{\partial}{\partial \rho}, \]

\[ X_9 = \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial \rho}. \]

For \( \gamma = 2 \) we add to (2) and (3) the operator

\[ X_{10} = t^2 \frac{\partial}{\partial t} + t x \frac{\partial}{\partial x} + t y \frac{\partial}{\partial y} + (x - tu) \frac{\partial}{\partial u} + (y - tv) \frac{\partial}{\partial v} - 4tp \frac{\partial}{\partial \rho} + 2tp \frac{\partial}{\partial \rho}. \]

The basic group for case A is denoted by \( G_7 \), while for case B it is denoted by \( G_9 \) for arbitrary \( \gamma \) and by \( G_{10} \) for \( \gamma = 2 \).

Table 1 gives the optimal system of one-parameter subgroups of group \( G_7 \).

The optimal system of one-parameter subgroups of group \( G_7 \) consists of operators 1-12 of Table 1 and the operators

\[ X_{1} + X_{4} + X_{7} + 2X_{9} + X_{10}, \]

\[ X_{1} + \alpha X_{4} + \beta X_{7} + \delta X_{9} + X_{10}. \]

(5)

The basic group for case A is denoted by \( G_7 \), while for case B it is denoted by \( G_9 \) for arbitrary \( \gamma \) and by \( G_{10} \) for \( \gamma = 2 \).

Table 2 gives the optimal system of two-parameter subgroups of group \( G_7 \); Table 4 does the same for group \( G_9 \) and subgroups 1-40 of Tables 3 and 4 do the same for group \( G_{10} \).

The form of the invariant solutions of rank unity is as follows. These solutions are derived from the two-parameter subgroups, \( U, V, P, \) and \( R \) are dependent on a single argument \( \lambda \), whose expressions in terms of \( t, x, \) and \( y \) vary with the subgroup and are given below. The necessary condition for an invariant solution is not obeyed for subgroups in which operator \( X_9 \) is one of the forming elements; moreover, the \( X_9 \) term in all subgroups affects only \( p \) and \( \rho \), and this effect is easily allowed for, so \( X_9 \) will not be considered. Also, \( I \) do not consider subgroups in which as one of the forming elements we have \( X_1, X_2, X_3, X_4, \) or \( X_5 \), since these give the stationary and one-dimensional case. For instance, the invariant solution for \( H = (X_6) \) takes the form

\[ u = U(t, x), \quad v = \frac{y}{t} + V(t, x), \]

\[ p = \frac{1}{t} P(t, x), \quad \rho = \frac{1}{t} R(t, x). \]

Then the \( H \) of system (1) is

\[ V_t + UV_x + tV = 0, \quad U_t + UU_x + R'P_x = 0, \]

\[ R_t + UR_x + RU_x = 0, \quad P_t + UP_x + A'U_x = 0 \]

with a known function \( U(t, x) \), so we have to deal with the solution of equations for one-dimensional motion.

For the subgroups of Table 2 we get invariant solutions of the form

\[ u = U, \quad u_x = V, \quad p = P, \quad \rho = R, \quad \lambda = \frac{t}{4} \].
Here r and \( \varphi \) are polar coordinates in the \((x, y)\) plane, while \( u_r \) and \( u_\varphi \) are the projections of the velocity on the axes of the polar coordinates.

For Table 3 we have

\[ 3. \ u = \frac{r^\beta}{1 + \beta^2} \ (U - iV + \lambda t), \]

\[ \lambda = \frac{t \varphi + \beta}{1 + \beta^2}, \]

\[ v = x + \frac{r^\beta}{1 + \beta^2} \bigl[ V + iU - \lambda t (t^2 + 2) \bigr], \]

\[ p = \frac{P}{1 + \beta^2}, \quad P = \frac{e^{-\varphi(t)} R}{(1 + t^2)^2}, \]

\[ \lambda = \frac{(x - y) e^{-\varphi(t)}}{1 + \beta^2}. \]

4. \( u = \frac{U - iV - 1/2 \beta \partial}{1 + \beta^2}, \)

\[ \lambda = \frac{(x - y) e^{-\varphi(t)}}{2}, \]

\[ \gamma = \frac{U - iV - 1/2 \beta \partial}{1 + \beta^2}, \]

\[ \lambda = \frac{e^{-\varphi(t)} R}{1 + \beta^2}. \]

6. To avoid complicating the formulas we consider the case \( \alpha = \beta = 0; \)

\[ u_r = \frac{r t}{1 + \beta^2} + \frac{U}{r}, \quad u_\varphi = \frac{V}{r}, \quad p = \frac{P}{1 + \beta^2}, \quad \rho = \frac{R}{1 + \beta^2}. \]

\[ \lambda = \frac{r}{V + 1 + \beta^2}. \]

7. Here also we assume \( \alpha = \beta = 0; \)

\[ u_r = \frac{r t}{1 + \beta^2} + \frac{r U}{1 + \beta^2}, \quad u_\varphi = \frac{r V}{1 + \beta^2}, \quad p = \frac{P}{1 + \beta^2}, \quad \rho = \frac{R}{1 + \beta^2}. \]

\[ \lambda = \frac{r}{V + 1 + \beta^2}. \]

### Table 3

<table>
<thead>
<tr>
<th>( X_1 + X_0 )</th>
<th>( a X_1 + X_1 )</th>
<th>( X_1 + \alpha X_7 + X_7 )</th>
<th>( X_1 + X_0 )</th>
<th>( X_1 + \alpha X_7 + X_7 )</th>
<th>( X_1 + X_0 )</th>
<th>( X_1 + X_2 + X_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha X_1 + X_0 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_0 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_0 )</td>
</tr>
<tr>
<td>( a X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( a X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( a X_1 + X_1 )</td>
</tr>
<tr>
<td>( a X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( a X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( a X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
<tr>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \beta X_1 + X_1 )</td>
<td>( \alpha X_1 + X_1 )</td>
</tr>
</tbody>
</table>