Energy bounds in dynamical problems for a semi-infinite magnetoelastic beam

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Abstract. The aim of this paper is to investigate the behaviour of the total energy of a magnetoelastic conductor occupying a semi-infinite prismatic cylinder in dynamical conditions. Precisely, we deduce some estimates for the energy \( W(x_3, t) \) of the portion of the medium at distance greater than \( x_3 \) from the base in terms of the data. First of all, we prove that the total energy \( W(0, t) \) is finite for all \( t > 0 \) provided \( W(0, 0) \) is finite. Then, using the first Korn inequality, we obtain that the estimate for \( W(x_3, t) \) depends only on the initial data if \( t \leq x_3/V \) (\( V \) = computable positive material constant); if \( t > x_3/V \) then the bound for \( W(x_3, t) \) depends on all the data of the problem.


Keywords. Magnetoelasticity.

1. Introduction

The purpose of this work is to investigate the behaviour of the total energy of a magnetoelastic conductor occupying a semi-infinite prismatic cylinder in dynamical conditions.

Here we consider a linear theory of magnetoelasticity (for infinitesimal deformations) in which only the ponderomotive force remains nonlinear in the presence of a magnetic field. For an exhaustive exposition of general theory of non-linear magnetoelasticity we refer to [1–2]. We suppose that the elastic medium under consideration is homogeneous, anisotropic and electrically conducting.

In particular we get some estimates for the energy of the portion of the medium at distance greater than \( x_3 \) from the base in terms of the initial and boundary data.

In our study we use arguments of the kind considered in [3–4] but, unlike the previous papers, we have to overcome the difficulty due to the fact that we assume the residual stress vanishing so that the elastic tensor \( C \) satisfies the minor symmetries too. This property of \( C \) implies that the positive definiteness condition for the elastic energy is defined only on the set of symmetric tensors. This difficulty can be overcome, as known, by using the first Korn inequality. We recall that this
inequality usually is available when the displacement vector $u$ vanishes on the boundary of the domain occupied by the solid.

In our case $u$ vanishes only on the lateral surface of the cylinder but, since this implies that it is not a rigid displacement, the first Korn inequality holds true in any cylinder of finite length [5, 6]. (In [6] the explicit dependence of the Korn constant on the dimensions of the region is given for particular domains.)

The paper is organized in the following way. After formulating in Section 2 four initial-boundary value problems, in Section 3 we obtain some useful properties of the cross-sectional work function and we show that the total energy is finite for all $t > 0$ if it is finite at the initial time. Sections 4, 5 are devoted to derive the evolution of the total energy with respect to both space and time. In fact we estimate through the data the energy $W \equiv W(x_3, t)$ stored in the cylinder beyond a distance $x_3$ from the base. More precisely, we show that if $t \leq x_3/V$ (where $V$ is a computable positive material constant) the bound for $W(x_3, t)$ depends only on the initial data; if $t \geq x_3/V$ the estimate of $W(x_3, t)$ depends on all the data of the problems.

2. Position of the problem in the framework of the classical linear theory

Let us consider a magnetoelastic conductor occupying a semi-infinite prismatic cylinder $\Omega = \{x \in \mathbb{R}^3; x' \equiv (x_1, x_2) \in D, x_3 \in \mathbb{R}^+ \equiv (0, +\infty)\}$ where $D$ is a bounded domain of $\mathbb{R}^2$ whose boundary $\partial D$ is sufficiently smooth to admit application of the divergence theorem.

In the framework of a linear theory in which only the ponderomotive force remains non-linear in the magnetic field ([1–2]), the dynamical equations for a homogeneous, anisotropic elastic conductor in $\Omega \times \mathbb{R}^+$ are the following:

\begin{align*}
\rho \ddot{u} &= \nabla \cdot T + J \times \mu H, \\
\varepsilon \dot{E} &= \nabla \times H - J, \\
\nabla \cdot \varepsilon E &= 0, \\
\nabla \cdot \mu H &= 0, \\
T &= C \cdot \nabla u = C \cdot e, \\
J &= \sigma (E + u \times \mu H),
\end{align*}

where $u$ is the displacement vector, $\rho$ is the constant mass density ($\rho > 0$), $T$ is the stress tensor, $J$ is the conduction current, $E$, $H$ are the electric, magnetic fields, $\varepsilon$, $\mu$ are the dielectric, magnetic permeability tensors, $C$ is the elastic tensor, $e$ is the infinitesimal strain tensor and $\sigma$ is the electric conductivity tensor.

Of course we suppose the external body force equal to zero in (1).

To system (1) we append the initial conditions

\begin{align*}
\dot{u}(x, 0) &= u_0(x), \\
\dot{v}(x, 0) &= v_0(x), \\
E(x, 0) &= E_0(x), \\
H(x, 0) &= H_0(x)
\end{align*}

$\forall x \in \Omega$.