SH-wave in a cylindrically anisotropic solid

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Abstract. Propagation of a transient SH-wave in a cylindrically anisotropic elastic solid is considered and an exact closed form solution for a special class of anisotropy is obtained. This is a case where a square root of the shear rigidity ratio is an integer. In this case we find an interesting wave pattern. However, the singularity at the wave front is of the same order as the one in the isotropic solid.

Keywords. Wave propagation, cylindrical anisotropy, SH-wave, exact solution, transient response, wave pattern.

Introduction

There are many kinds of anisotropic solids. Among them, naturally or artificially created media show cylindrical anisotropy to mechanical loading. Wood and fiber wound pipes are typical ones. The elastodynamic analysis for this cylindrically anisotropic solid is less carried out than that for the orthotropic and transversely isotropic solids [1, 2]. A few results on the wave propagation in the cylindrically anisotropic solid are known. Shul'ga and Ramskaya [3–5] have considered the propagation of harmonic waves in a cylindrically anisotropic hollow cylinder and have obtained the dispersion curves. Bostrom, Johansson and Svedberg [6] have considered the propagation of an SH wave in the cylindrically anisotropic solid and have found interesting wave patterns numerically with superposing harmonic waves. All of the above results are numerical not fully analytical. To the authors’ knowledge, there are no analytical solutions for the transient problem in the cylindrically anisotropic solid.

In the present paper we have considered the propagation of a transient SH-wave in a cylindrically anisotropic elastic solid and have obtained an exact closed form solution, that is a Green function for a point source. However, the present closed form solution is applicable only to a special class of cylindrical anisotropy, i.e. the square root of the rigidity ratio, \( \lambda = \sqrt{\mu_{\theta}/\mu_r} \), must be an integer. For the other case of \( \lambda \), the solution is not valid. Thus, for more general cases of the cylindrical anisotropy, an other solution method has to be explored. So, in spite of...
the incompleteness of the present paper, the authors present only one exact closed form solution so that more attention would be paid to the dynamic problem of the cylindrically anisotropic solid.

Formulation

Let us consider a cylindrically anisotropic elastic solid and assume an antiplane deformation. Taking polar coordinates as in Fig. 1, we have governing equations given by

\[
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} = \rho \frac{\partial^2 u_z}{\partial t^2} - \rho B(r, \theta, t),
\]

\[
\sigma_{rz} = \mu_r \frac{\partial u_z}{\partial r}, \quad \sigma_{\theta z} = \mu_\theta \frac{\partial u_z}{\partial \theta}
\]

where \(\rho\) is density, and \(\mu_r\) and \(\mu_\theta\) are shear rigidities in radial and circumferential directions, respectively. Body force \(B(r, \theta, t)\) is assumed as a source function and is placed at a point apart from the origin. That is

\[
B(r, \theta, t) = B_0 \delta(r - a)\delta(\theta)\delta(t),
\]

where \(\delta(.)\) is Dirac's delta function.

Substituting Eqs. (2) and (3) into Eq. (1), we have a simple wave equation for \(u_z\)

\[
\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \left(\frac{\lambda}{r}\right)^2 \frac{\partial^2 u_z}{\partial \theta^2} - \frac{1}{c^2} \frac{\partial^2 u_z}{\partial t^2} = -\frac{B_0}{c^2} \delta(r - a)\delta(\theta)\delta(t),
\]

where \(\lambda\) is the square root of rigidity ratio and \(c\) is the velocity of SH-wave in the radial direction. They are defined by

\[
\lambda = (\mu_\theta/\mu_r)^{1/2}, \quad c = (\mu_r/\rho)^{1/2}.
\]