Existence Theorems for Abstract Multidimensional Control Problems

LAMBERTO CESARI

Abstract. In the present paper, the author discusses an abstract formulation of control problems involving general operators \( \mathcal{L} : S \rightarrow V \), \( \mathcal{M} : S \rightarrow Y \) from a Banach space \( S \) into space \( V \) and \( Y \) of vector functions in a fixed domain with components in \( L_p \), \( p \geq 1 \). For this general formulation, the author states closure theorems, lower closure theorems, and existence theorems for an optimal solution. It is then shown that the problems of control involving Dieudonné–Rashevski partial differential equations previously considered by the author are particular cases of the present formulation. Finally, it is shown by examples that problems of control involving usual partial differential equations, linear or not, as well as other functional relations, can be framed in the present formulation. The present work concerns problems with distributed controls. Work concerning problems with distributed as well as boundary controls is forthcoming.

1. Introduction

We present here existence theorems for multidimensional optimal control problems in an abstract setting, which are extensions of theorems proved in a concrete form in previous papers (Refs. 1–4). The present formulation for general Lagrange problems includes also a number of results which have appeared before for free problems only (Refs. 5–11). The present formulation concerns only distributed control problems in the terminology of Lions (Ref. 12). Extensions including boundary control problems will be discussed elsewhere.

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1 Paper received December 19, 1969; in revised form, January 29, 1970. Parts of this paper were read at the International Conference of Optimal Control, Tbilisi, Georgia, USSR, 1969, and at the Conference on Optimal Control, Ann Arbor, Michigan, 1969 (Tenth Annual Meeting of the Society for Natural Philosophy). This research was partially supported by AFOSR Research Project No. 69-1662.

2 Professor of Mathematics, University of Michigan, Ann Arbor, Michigan.
We are interested in control problems where the state variable is an element of a Banach space \( S \) with norm \( \| x \| \), where \( \mathcal{L} : S \to V \) and \( \mathcal{U} : S \to Y \) are two operators, \( \mathcal{L} \) possibly unbounded, \( V \) and \( Y \) functions spaces of elements \( y(t) = (y_1, \ldots, y_s), \ v(t) = (v_1, \ldots, v_r) \), \( t \in G \), and where the controls are also vector functions \( u(t) = (u_1, \ldots, u_m), \ t \in G, \ G \) a bounded open subset of \( E_v, \ v \geq 1 \). Thus, we consider control problems monitored by a functional equation of the form

\[
(\mathcal{L}x)(t) = f(t, (\mathcal{U}x)(t), u(t)) \quad \text{a.e. in } G,
\]

with usual constraints

\[
(t, (\mathcal{U}x)(t)) \in A, \quad u(t) \in U(t, (\mathcal{U}x)(t)) \quad \text{a.e. in } G,
\]

and functional

\[
I[x, u] = \int_G f_0(t, (\mathcal{U}x)(t), u(t)) \, dt.
\]

Details and more general formulations will be indicated below. Whenever \( S \) is a space of vector functions on \( G \) and \( \mathcal{L} \) and \( \mathcal{U} \) are differential operators, then (1) reduces to a usual differential system in \( G \).

2. Abstract Functional Equation

Let \( G \) be a given open bounded subset of the \( t \)-space \( E_v, \ t = (t_1, \ldots, t_v), \ v \geq 1 \), let \( Y \) be a space of \( s \)-vector functions \( y(t) = (y_1, \ldots, y_s), \ t \in G \), whose components \( y_i \) are \( L_{p_i} \)-integrable in \( G, \ p_i \geq 1, \ i = 1, \ldots, s \), and let \( V \) be a space of \( r \)-vector functions \( v(t) = (v_1, \ldots, v_r), \ t \in G \), whose components \( v_j \) are \( L_{p_j} \)-integrable in \( G, \ p_j \geq 1, \ j = 1, \ldots, r. \) Thus, \( Y \subset L', \ V \subset L'' \), where \( L' = \bigcap_{i=1}^s L_{p_i}(G), \ L'' = \bigcap_{j=1}^r L_{p_j}(G) \). We shall take in \( Y \) and \( V \) the usual norms

\[
\| y \| = \sqrt{\sum_{i=1}^s \| y_i \|_{L_{p_i}}^2}, \quad \| v \| = \sqrt{\sum_{j=1}^r \| v_j \|_{L_{p_j}}^2},
\]

or equivalent ones, and we shall denote them also by \( \| y \|_p \), or \( \| y \|_{L_p} \). Let \( T \) be the space, or set, of all \( m \)-vector functions \( u(t) = (u_1, \ldots, u_m), \ t \in G \), whose components \( u_i \) are measurable in \( G \).

Let \( S \) be a Banach space of elements \( x \) and norm \( \| x \| \), and let

\[
\mathcal{U} : S \to Y, \quad \mathcal{L} : S \to V
\]