On the Existence of Optimal Controls

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Abstract. A control process described by Eq. (1) is considered. Existence theorems for controls which minimize a functional of a general type are given by using weak compactness criteria.

1. Introduction

In the present paper, we consider control processes described by an integral equation of the form

\[
x(t, u, v) = T_t(u, v) + \int_{t_0}^{t} f(s, x(s, u, v), v, u(s)) \, ds, \quad t \in I,
\]

where \( t \) is the time variable, \( t \in I = [t^0, t^1] \subset [T_0, T_1] \), \( V \) is a normed linear space, \( T_t \) is a continuous linear operator from a subset of \( L_{m,p}(I) \times V \) into \( \mathbb{R}^n \), and \( f \) is a function from \( [T_0, T_1] \times \mathbb{R}^n \times V \times \mathbb{R}^m \) into \( \mathbb{R}^n \). The pair \((u, v)\) consisting of a function \( u: t \rightarrow u(t) \in \mathbb{R}^n \) defined on an interval \( I_u \) contained in \( [T_0, T_1] \) and of an element \( v \in V \) is said to be a control of (1) if there exists a unique absolutely continuous function \( x: t \in I_u \rightarrow x(t, u, v) \) which satisfies (1). Such a function will be called a response. All such controls satisfying a constraint of the type

\[
H(u(t), x(t, u, v), v, t) \leq 0, \quad t \in I_u,
\]

where \( H \) is a function from \( \mathbb{R}^m \times \mathbb{R}^n \times V \times [T_0, T_1] \) into \( \mathbb{R} \), are called

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admissible controls. Such constraint sets have been considered by Poljak in Ref. 1.

Problems of controllability and existence of optimal controls have been studied under various hypotheses by many authors. In addition, the function \( f \) independent of \( v \) is assumed to be continuous and continuously differentiable with respect to \( x \) by Filippov in Ref. 2 and by Lee and Markus in Ref. 3, only continuous by Cesari in Ref. 4 and by Stoddart in Ref. 5, and satisfying the Caratheodory conditions by Gossez in Ref. 6. Furthermore, \( f \) is assumed to be linear in \( x \) by Neustadt in Ref. 7, linear in \( x \) and in \( u \) by Antosiewicz in Ref. 8, by Conti in Ref. 9, and by Jacobs in Ref. 10. Our case deals with a function \( f(t, x, v, u) \) which satisfies the Caratheodory hypotheses, is Lipschitzian in \( x \), linear in \( u \), and uniformly continuous in \( v \).

The plan of this paper is the following. Section 2 contains notations and basic definitions. In Section 3, we introduce the single-valued map

\[
S : (u, v) \rightarrow x(\cdot, u, v),
\]

which corresponds with each control \((u, v)\) its response and prove that \( S \) satisfies a suitable continuity property. In Section 4, we give some compactness criteria for the set \( Q' \) of all the admissible controls. It should be noted that the definition of \( Q' \) involves not only \((u, v)\) but also the response \( x \), so that in proving the compactness of \( Q' \) we make use of the continuity of \( S \). Finally, in the last section, we consider a real functional

\[
\Phi : Q' \rightarrow R
\]

and prove that, under a weak convexity condition, \( \Phi \) may be optimized.

2. Notations and Preliminaries

Let \( R^h \) be the real Euclidean \( h \)-dimensional space with the norm

\[
|x| = \sum_{i=1}^{h} |x_i|, \quad x = (x_1, x_2, ..., x_h).
\]

For each \( h \times h \)-matrix \( A = (a_{ij}) \), the norm is given by

\[
|A| = \sum_{i,j=1}^{h} |a_{ij}|.
\]