Control-Space Properties of Cooperative Games

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Abstract. If two or more players agree to cooperate while playing a game, they help one another to minimize their respective costs as long as it is not to their individual disadvantages. This leads at once to the concept of undominated solutions to a game. An undominated or Pareto-optimal solution has the property that, compared to any other solution, at least one player does worse or all do the same if they use a solution other than the Pareto-optimal one.

Closely related to the concept of a Pareto-optimal solution is the absolutely cooperative solution. Such a solution has the property that, compared to any other permissible solution, every player does no better if a solution other than the absolutely cooperative one is employed.

This paper deals with control-space properties of Pareto-optimal and absolutely cooperative solutions for both static, continuous games and differential games. Conditions are given for cases in which solutions to the Pareto-optimal and absolutely cooperative games lie in the interior or on the boundary of the control set.

The solution of a Pareto-optimal or absolutely cooperative game is related to the solution of a minimization problem with a vector cost criterion. The question of whether or not a problem with a vector cost criterion can be reduced to a family of minimization problems with a scalar cost criterion is also discussed.

An example is given to illustrate the theory.

1. Introduction

According to Von Neumann and Morgenstern (Ref. 1, p. 49), a game is the totality of rules which describes it, and hence an abstraction. Rules are
simply a set of instructions to the players on how to play the game within this context. We shall assume that the rules, in addition to prescribing the cost function for each player, the system which the players are manipulating, and the limitations on their controls, also includes information which sets the mood of the game by requiring that the players cooperate (up to the point of disadvantage) or not cooperate (each attempts to minimize his own cost with no consideration for others), etc. We shall assume here that the players are rational and will play only according to the rules of the game.

These latter requirements give character to the game. We may expect that solutions exist under certain moods of play and do not exist under others. Three main moods which may be considered as part of the objectives are of interest.

1.1. Nash Equilibrium (Ref. 2, p. 287). Here, the absence of coalitions is assumed, with each player acting independently, without collaboration or communication with any of the other players. The concept of an equilibrium solution is introduced. The equilibrium property is such that the equilibrium solution is secure against any attempt by one player unilaterally to alter his strategy. That is, assuming that every player is using his Nash control, if a given player plays non-Nash-optimally, he will do no better, and similarly for every other player.

1.2. Minimax (or Security) Solutions (Ref. 3, p. 190). In this case, each player is to assume that all the others are out to get him. A minimax solution for a given player is one such that he will do no better if he plays non-minimax-optimally (assuming that all players are using this solution), and he will do no worse if any one of the other players plays non-minimax-optimally.

1.2.1. Absolutely Conflicting Solution. If all players’ minimax solutions are the same, then such a solution may be termed an absolutely conflicting one; in fact, it is a Nash equilibrium.

1.3. Pareto-Optimal (or Noninferior) Solution (Ref. 4, p. 118). Here, cooperation is assumed in the sense that each player helps the others up to the point of disadvantage to himself. A Pareto-optimal solution is characterized by the fact that, if one of the players plays non-Pareto-optimally, then at least one of the players (not necessarily the same one) does worse or all do the same.