SURVEY PAPER

Multiplier and Gradient Methods

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Abstract. The main purpose of this paper is to suggest a method for finding the minimum of a function \( f(x) \) subject to the constraint \( g(x) = 0 \). The method consists of replacing \( f \) by \( F = f + \lambda g + \frac{1}{2}cg^2 \), where \( c \) is a suitably large constant, and computing the appropriate value of the Lagrange multiplier. Only the simplest algorithm is presented. The remaining part of the paper is devoted to a survey of known methods for finding unconstrained minima, with special emphasis on the various gradient techniques that are available. This includes Newton's method and the method of conjugate gradients.

1. Introduction

About twenty years ago, the author became interested in computational methods for optimal control problems (Ref. 1). This interest was stimulated by an attempt to compute the time-optimal path of an airplane from take-off to level flight at a prescribed position and velocity. At that time, large-scale digital-computing machines were not available. Computing had to be carried out by analog computers or by mechanical desk computers. In Ref. 2, an attempt was made to compute the time-optimal path for an airplane by integrating the corresponding Euler-Lagrange equations on an analog computer (REAC). However, the differential equations were unstable and the results were unsatisfactory. However, a good estimate could be found by

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hand computation using special properties of the problem. This experience convinced the author that general procedures should be devised for obtaining solutions or for improving estimates of solutions. Accordingly, the author experimented with three methods, namely, Newton’s method, the gradient method, and the method of penalty functions. Since he was restricted to the use of hand computation, the author considered only simple variational problems which possessed nonminimizing as well as minimizing extremals. It was found that Newton’s method and the gradient method were very effective (Refs. 3–5).

The author, however, had difficulties with the method of penalty functions because of round-off errors. To obtain any accuracy to the solution of the problem considered required carrying more significant figures than were convenient in hand computation. Although the method of penalty functions has been used with reasonable success in recent years, the author has always felt that an improvement of the method could be made. The purpose of this paper is to suggest a modification of the method of penalty functions which we shall call the method of multipliers. In addition, we shall make some remarks concerning Newton’s method, the method of gradients, and conjugate gradients that may be useful.

2. Constrained and Unconstrained Minima

Before describing the method of multipliers, it is instructive to recall a connection between constrained and unconstrained minima upon which the method is based. We shall consider only the simplest case, in which a point \( x_0 \) affords a minimum to a real-valued function \( f(x) = f(x_1, \ldots, x_m) \) subject to a single constraint

\[
g(x) = 0
\]

The extension to the case in which \( g \) is vector-valued is immediate. We assume that \( f \) and \( g \) are of class \( C^n \) and that the gradient

\[
g'(x) = (\partial g(x)/\partial x^i)
\]

of \( g \) is not zero at \( x_0 \). Then, there exists a multiplier \( \lambda \) such that, if we set \( G = f + \lambda g \), we have

\[
G'(x_0) = 0, \quad g(x_0) = 0 \quad (2)
\]

\[
G''(x_0, h) = \sum_{i,j=1}^{m} (\partial^2 G/\partial x^i \partial x^j) h^i h^j \geq 0 \quad (3)
\]