EXPANSION OF THE MUTUAL DISTANCE BETWEEN TWO
PLANETS RAISED TO ANY NEGATIVE POWER

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Abstract. An expression for $\Delta^{-s}$ in terms of the Poincaré variables $L, \lambda, H, K, P, Q$ has been evaluated. The inclinations of the two planets are referred to a common fixed plane. We neglect in the final formula powers higher than the third of the Poincaré variables.

1. Introduction

In a previous paper (Kamel and Bakry, 1981), we obtained the expression for $\Delta^{-s} = \rho \rho' (1 + a^2 - 2a \cos \theta)^{-s/2}$, the mutual distance between two bodies raised to negative natural integers, in terms of the true anomalies, and the classical elliptical orbital elements, referring to a common fixed plane and expanding up to the fourth power of eccentricities and sines of inclinations. The analysis yielded two cases, the first when $\gamma' > \gamma$, the second when $\gamma > \gamma'$.

In our present paper we proceed to acquire the expression for $\Delta^{-s}$, firstly in terms of the mean anomalies and classical orbital elements, secondly in terms of the Poincaré variables, expanding up to the third power of these variables, that means up to the third power of the eccentricities and sines of inclinations.

Although we aim to obtain the final expression up to terms of the third power of $H, K, P, Q$, numerous expansions involved in the analysis were established up to the fourth power of the eccentricities and inclinations, which indeed multiplied the labour.

The splitting into two cases with two corresponding final formulas does not occur when we expand up to the third power of $e, \gamma$. It arises only when we extend our expansions to powers higher than the third with respect to $e, \gamma$.

2. Expression for $\sin \left( \frac{j_1 \varphi + j_2 \varphi'}{2} \right)$

Equality (2) of our previous paper (Kamel and Bakry, 1981) is expressed in terms of the mutual inclination of the two orbital planes. We rewrite this equality referring the inclinations of the two planets to a common fixed plane, the ecliptic for instance, instead of using the mutual inclination $I$. To carry out this transformation, we utilize the formula (cf. Kamel, 1970)

$$\sin^2 \frac{I}{2} = \frac{1}{4} [\gamma^2 + \gamma'^2 - 2\gamma \gamma' \cos (\Omega - \Omega')] + \frac{1}{16} (\gamma^2 - \gamma'^2)^2.$$  \hspace{1cm} (1)

Then we have
\[ \Delta_s = \rho^D \rho^{-D-s}(1 + \alpha^2 - 2\alpha \cos \Theta)^{-s/2} = \]
\[ = \frac{1}{\alpha} \sum_{j=-\infty}^{\infty} \left[ b^{(j)}_{s/2} + e^2 \left( \frac{s}{2} + \frac{3}{4} \right) D + \frac{s^2}{4} \right] b^{(j)}_{s/2} + e^2 \left( \frac{s}{2} - \frac{3}{4} \right) D b^{(j)}_{s/2} + \]
\[ + e^4 \left( \left( \frac{5}{8} + \frac{5}{8} s \right) D^3 + \left( \frac{1}{8} s^2 + \frac{1}{8} s + \frac{1}{8} \right) D^2 + \left( \frac{3}{16} + \frac{1}{16} s^2 + \frac{1}{16} s + \frac{1}{16} \right) D \right) + \]
\[ + \frac{s^4}{64} + \frac{5}{8} s^3 + \frac{3}{4} s \right] b^{(j)}_{s/2} + e^2 e^2 \left( \frac{s}{16} D^4 + \frac{s^2}{8} D^3 + \left( \frac{s^2}{16} - \frac{s}{16} \right) D^2 \right) + \]
\[ + \left( -\frac{s^2}{16} - \frac{3}{16} D^3 \right) b^{(j)}_{s/2} + e^2 \left( \frac{s}{16} D^4 - \frac{s}{16} D^3 + \frac{s}{16} D^2 - \frac{s}{16} D \right) b^{(j)}_{s/2} - \]
\[ - \left( \frac{4}{16} \gamma_1^2 + \frac{4}{16} \gamma_2^2 - \frac{1}{16} \gamma_1 \gamma_2 \cos (\Omega_1 - \Omega_2) + \frac{4}{16} (\gamma_1^2 + \gamma_2^2 - 2 \gamma_1 \gamma_2) \right) s \frac{\alpha^2}{1 - \alpha^2} b^{(j)}_{s/2} - \]
\[ - \left( \frac{4}{16} \gamma_1^2 + \frac{4}{16} \gamma_2^2 - \frac{1}{16} \gamma_1 \gamma_2 \cos (\Omega_1 - \Omega_2) \right) \frac{\alpha^2}{1 - \alpha^2} \left( \frac{1}{16} D^2 + \frac{1}{16} D^3 + \frac{1}{16} D^2 \right) + \]
\[ + \frac{1}{16} \gamma_1^2 \gamma_2^2 + \frac{1}{16} \gamma_1 \gamma_2 \cos (\Omega_1 - \Omega_2) - \frac{1}{16} \gamma_1 \gamma_2 \cos (\Omega_1 - \Omega_2) \right) \times \]
\[ \times \left( \frac{\alpha^2}{1 - \alpha^2} \left( -\frac{1}{16} D^2 + \frac{1}{16} D^3 \right) b^{(j)}_{s/2} + \left( \frac{s}{16} \gamma_1^2 + \frac{s}{16} \gamma_2^2 + \frac{s}{16} \gamma_1 \gamma_2 \right) \frac{\alpha^2}{1 - \alpha^2} \left( s D + s^2 \right) b^{(j)}_{s/2} \right) \times \]
\[ \times \cos \left( \phi - \phi' + \pi - \pi' \right) + \left[ e^2 (D + s) b^{(j)}_{s/2} + e^3 \left( \frac{1}{8} D^3 + \left( \frac{1}{8} s + \frac{1}{8} \right) D^2 \right) + \right. \]
\[ + \left( \frac{1}{8} s^2 + \frac{1}{8} s + \frac{1}{8} \right) D + \left. \left( \frac{1}{8} s^3 + \frac{1}{8} s^2 + \frac{1}{8} s \right) b^{(j)}_{s/2} + e^2 e^2 \left( \frac{1}{8} D^3 + \left( \frac{1}{8} s - \frac{1}{8} \right) D^2 - \frac{1}{8} s D \right) b^{(j)}_{s/2} - \right. \]
\[ - \left( \frac{1}{8} e \gamma_1^2 + \frac{1}{8} e \gamma_2^2 - \frac{1}{8} e \gamma_1 \gamma_2 \cos (\Omega_1 - \Omega_2) \right) \frac{\alpha^2}{1 - \alpha^2} \left( s D + s^2 \right) b^{(j)}_{s/2} \right) \times \]
\[ \times \cos \left( j \varphi + (1 - j) \varphi' + j \pi - j \pi' \right) + \left\{ -e D b^{(j)}_{s/2} + e e^2 \left( -\frac{1}{4} D^3 + \left( -\frac{1}{2} s - \frac{1}{2} \right) D \right) + \right. \]
\[ + \left. \left( -\frac{1}{8} s^2 + \frac{1}{8} s \right) D) b^{(j)}_{s/2} + e^3 \left( -\frac{1}{8} D^3 + \frac{1}{8} D^2 - \frac{1}{8} D \right) b^{(j)}_{s/2} + \right. \]
\[ + \left( \frac{1}{8} e \gamma_1^2 + \frac{1}{8} e \gamma_2^2 - \frac{1}{8} e \gamma_1 \gamma_2 \cos (\Omega_1 - \Omega_2) \right) \frac{\alpha^2}{1 - \alpha^2} s D b^{(j)}_{s/2} \right) \times \]
\[ \times \cos \left( (1 + j) \varphi - j \varphi' + j \pi - j \pi' \right) + \left\{ e^2 \left( \frac{1}{4} D^2 + \left( \frac{1}{4} s - \frac{1}{4} \right) D \right) + \frac{1}{4} s^2 - \frac{1}{4} s \right) b^{(j)}_{s/2} + \right. \]
\[ + e^4 \left( \frac{1}{8} D^4 + \left( \frac{1}{8} s + \frac{1}{8} \right) D^3 + \left( \frac{1}{8} s^2 + \frac{1}{8} s - \frac{1}{8} D^2 + \left( \frac{1}{8} s^3 + \frac{1}{8} s^2 - \frac{1}{8} s - \frac{1}{8} \right) D + \right. \]
\[ + \left. \frac{1}{8} s^4 + \frac{1}{8} s^3 - \frac{1}{8} s - \frac{1}{8} b^{(j)}_{s/2} + e^2 e^2 \left( \frac{1}{16} D^4 + \left( \frac{1}{16} s - \frac{1}{16} \right) D^3 + \left( \frac{1}{16} s^2 - \frac{1}{16} s + \frac{1}{16} \right) D \right) + \right. \]