Variational Problems in Space Flight Mechanics

G. L. Grodzovsky

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Abstract. The effect of weight limitations on the parameters of the optimum motion of a variable-mass body in a gravitational field is considered. An analysis is made of the optimum motion for three possible propulsion systems: limited exit velocity, limited power, and limited thrust. Solutions are presented for some typical cases. The problems of selecting the optimum weight parameters, finding the optimum propulsion system controls, and determining the optimum trajectory are solved simultaneously.

1. Introduction

This paper considers the simultaneous problems of selecting the optimum weight parameters, finding the optimum propulsion system controls, and determining the optimum trajectory of a space vehicle. The following limitations imposed on the propulsion system are important: limited exit velocity, limited power, and limited thrust (Ref. 1). Normally, the optimum motion requires that the limiting values of the controls be used.

The effects of weight limitations on the propulsion system parameters were previously investigated for a limited-power system (Refs. 1–5). Here, the same effects are considered for all the cases mentioned above.

The effect of the propulsion system weight and the structural factor on the optimum motion of a variable-mass body whose engine has a limited exit velocity is considered in detail. It is shown that, even for a low engine specific weight, the consideration of the propulsion system weight affects the optimum motion considerably. This leads to the extremal problem of determining that engine weight which ensures maximum payload. The total variational problem

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2 Professor, Moscow Physical-Technical Institute, Moscow, USSR, and Editor, Referativnyi Zhurnal, Mekhanika, Academy of Sciences of the USSR, Moscow, USSR.
is separated into a dynamic problem and a weight problem. The dynamic problem is reduced to the well-known problem of the optimum motion with an ideally weightless engine having limited thrust. The algorithm for the transition to the solution of the total variational problem is derived; this algorithm allows for the propulsion system weight (which depends on the maximum thrust) and the structural weight (which depends on the fuel weight).

Examples are presented for the following typical problems: vertical climb in a constant gravitational field, vertical landing in a constant gravitational field, disorbiting from a circular orbit, and escape from a circular orbit.

2. Variational Problems with Limited Exit Velocity

The object of space flight mechanics is to determine the optimum weight parameters, the optimum propulsion system controls, and the optimum flight trajectory. These topics were investigated for low-thrust vehicles: these are vehicles with a continuously operated propulsion system having limited power and weight proportional to the launch weight (Refs. 1–5). The considerable effect of the propulsion system weight on the optimum motion of a variable-mass body whose engine has limited exit velocity and low specific weight was shown in Ref. 6. The optimization of this motion considering the simultaneous effect of the propulsion system weight and the structural factor is investigated below.

Among propulsion systems having limited exit velocity, we mention the thermal rocket engine. For this engine, the exit velocity $V$ is limited by the intensity of the chemical reaction and the allowable exhaust temperature. This propulsion system is characterized by an approximately constant maximum exit velocity $V_{\text{max}}$ and a comparatively low specific weight $\gamma = G_x/P_{\text{max}}$, of the order of $\gamma \approx 0.02–0.05$ (Ref. 1); here, $G_x$ is the propulsion system weight and $P_{\text{max}}$ is the maximum thrust. As shown in Ref. 6, low values of the parameter $\gamma$ are desirable for maneuvers in which the engine operating time $t_\text{e}$ is a large fraction of the total maneuver time $T$.

Consider the problem of maximizing the payload $G_x$ delivered in the time $T$ between two points of the phase space $r_0$, $\dot{r}_0$ and $r_1$, $\dot{r}_1$ with an engine having limited thrust $P \leq P_{\text{max}}$ and limited exit velocity $V \leq V_{\text{max}}$. Here, this problem is investigated considering the propulsion system weight and the structural factor.

As fuel is consumed, the rate of decrease in vehicle weight is given by the relation

$$\dot{G}_x = -gP/V$$

(1)