Sufficiency Theorems for Optimal Control\textsuperscript{1}

GEORGE LEITMANN\textsuperscript{2}

Abstract. Two sufficiency theorems, related to those of Kalman, Athans and Falb, and Boltyanskii, are derived. The derivation is based on the invariance of a certain line integral along a solution curve reaching the terminal state set. The theorems are applied to some simple problems.

1. Problem Statement

We consider a system with state equation

\begin{equation}
\dot{x}(t) = f(x(t), u(t))
\end{equation}

where \(x\) and \(f\) are \(n\)-vectors and \(u\) is an \(m\)-vector. The control \(u\) belongs to the set of admissible controls, here the set of bounded, piecewise continuous functions of \(t\) (on a bounded interval) whose values have range in a given set \(U \subseteq E^m\); the set \(U\) may be state-dependent. The functions \(f_i\) and \(\partial f_i/\partial x_j\), \(i, j = 1, 2, \ldots, n\), are continuous on \(E^n \times U\).

The state is to be transferred from \(x(t_0) = x^0\) to \(x(t_1)\in \theta^1\), where \(\theta^1\) is a given set in \(E^n\); the interval \(t_1 - t_0\) is not specified.\textsuperscript{3} The performance index

\begin{equation}
\int_{t_0}^{t_1} f_\theta(x(t), u(t)) \, dt
\end{equation}

is to be minimized, where \(f_\theta\) has the same properties as \(f\). A control that accomplishes this task with respect to a given set of admissible controls is said to be optimal with respect to that set.

\textsuperscript{1} Paper received February 2, 1968. This work was supported by the Office of Naval Research, Contract No. Nonr 3656 (31).

\textsuperscript{2} Professor of Engineering Science, University of California at Berkeley, Berkeley, California.

\textsuperscript{3} The case of nonautonomous state equation and time-dependent end conditions can be treated in the usual manner by introducing \(x_{n+1} = t\).
2. Sufficiency Theorem 1

Here, we give a theorem that is similar to one of Kalman (Ref. 1) and of Athans and Falb (Ref. 2). Let $X$ be an open set in $E^n$. Then, we have the following theorem:

**Theorem 1.** The control $u^*(t)$ with corresponding solution $x^*(t)$, $t_0 \leq t \leq t_1^*$, is optimal with respect to all the admissible controls $u(t)$ with corresponding solutions $x(t)$, $t_0 \leq t \leq t_1$, such that $x(t_1) \in \theta^1$ and $x(t) \in X \forall t \in [t_0, t_1]$, if there exists a scalar function $V^*$ of class $C^1$ on $X$ with $\lim_{x \to x_0} V^*(x) = 0$ for $x \in \theta^1$ and $x \in \partial X$, where $\partial$ is a solution curve of (1) generated by an admissible control, such that

(i) $f_0(x^*(t), u^*(t)) + \nabla V^*(x^*(t)) \cdot f(x^*(t), u^*(t)) = 0 \forall t \in [t_0, t_1^*]$ and

(ii) $f_0(x, u) + \nabla V^*(x) \cdot f(x, u) \geq 0 \forall u \in U, \forall x \in X$

**Proof.** Consider a solution curve $p$ of (1) from $x^0$ to $\theta^1$, generated by an admissible control $u(t)$, $t_0 \leq t \leq t_1$, such that the solution $x(t) \in X \forall t \in [t_0, t_1]$. As a consequence of the hypotheses of the theorem,

$$I_p \triangleq \lim_{y \to x_0(t_1)} \int_{x_0}^y \nabla V^*(x) \cdot dx = -V^*(x_0) \tag{3}$$

By (i) of the theorem, together with (1) and (3),

$$\int_{t_0}^{t_1^*} f_0(x^*(t), u^*(t)) \, dt = V^*(x_0) \tag{4}$$

Consider the difference in the value of the payoff

$$\Delta \triangleq \int_{t_0}^{t_1} f_0(x(t), u(t)) \, dt - \int_{t_0}^{t_1^*} f_0(x^*(t), u^*(t)) \, dt \tag{5}$$

Now, add the null term $I_p + V^*(x_0)$ to the right-hand side of (5). In view of (1),

$$I_p = \int_{t_0}^{t_1} \nabla V^*(x(t)) \cdot f(x(t), u(t)) \, dt \tag{6}$$

---

4 That is, $\theta^1 \cap X$ may belong to $\partial X$. 