Sufficiency Theorems for Optimal Control

GEORGE LEITMANN

Abstract. Two sufficiency theorems, related to those of Kalman, Athans and Falb, and Boltyanskii, are derived. The derivation is based on the invariance of a certain line integral along a solution curve reaching the terminal state set. The theorems are applied to some simple problems.

1. Problem Statement

We consider a system with state equation

\[ x(t) = f(x(t), u(t)) \]  

where \( x \) and \( f \) are \( n \)-vectors and \( u \) is an \( m \)-vector. The control \( u \) belongs to the set of admissible controls, here the set of bounded, piecewise continuous functions of \( t \) (on a bounded interval) whose values have range in a given set \( U \subset E^m \); the set \( U \) may be state-dependent. The functions \( f_i \) and \( \partial f_i / \partial x_j, \)

\( i, j = 1, 2, \ldots, n \), are continuous on \( E^n \times \bar{U} \).

The state is to be transferred from \( x(t_0) = x^0 \) to \( x(t_1) \in \theta^1 \), where \( \theta^1 \) is a given set in \( E^n \); the interval \( t_1 - t_0 \) is not specified. The performance index

\[ \int_{t_0}^{t_1} f_0(x(t), u(t)) \, dt \]  

is to be minimized, where \( f_0 \) has the same properties as \( f \). A control that accomplishes this task with respect to a given set of admissible controls is said to be optimal with respect to that set.
2. Sufficiency Theorem 1

Here, we give a theorem that is similar to one of Kalman (Ref. 1) and of Athans and Falb (Ref. 2). Let $X$ be an open set in $E^n$. Then, we have the following theorem:

**Theorem 1.** The control $u^*(t)$ with corresponding solution $x^*(t)$, $t_0 \leq t \leq t_1^*$, is optimal with respect to all the admissible controls $u(t)$ with corresponding solutions $x(t)$, $t_0 \leq t \leq t_1$, such that $x(t_1) \in \theta^1$ and $x(t) \in X \ \forall t \in [t_0, t_1)$, if there exists a scalar function $V^*$ of class $C^1$ on $X$ with $\lim_{x \to \partial X} V^*(x) = 0$ for $x \in \theta^1$ and $x \in p \subset \bar{X}$, where $p$ is a solution curve of (1) generated by an admissible control, such that

1. $f_0(x^*(t), u^*(t)) + \nabla V^*(x^*(t)) \cdot f(x^*(t), u^*(t)) = 0 \ \forall t \in [t_0, t_1^*)$

and

2. $f_0(x, u) + \nabla V^*(x) \cdot f(x, u) \geq 0 \ \forall u \in U, \ \forall x \in X$

**Proof.** Consider a solution curve $p$ of (1) from $x^0$ to $\theta^1$, generated by an admissible control $u(t)$, $t_0 \leq t \leq t_1$, such that the solution $x(t) \in X$ $\forall t \in [t_0, t_1)$.\(^4\) As a consequence of the hypotheses of the theorem,

\[
I_p \triangleq \lim_{y \to x^0} \int_{x^0}^y \nabla V^*(x) \cdot dx = -V^*(x^0) \quad (3)
\]

By (i) of the theorem, together with (1) and (3),

\[
\int_{t_0}^{t_1} f_0(x^*(t), u^*(t)) \ dt = V^*(x^0) \quad (4)
\]

Consider the difference in the value of the payoff

\[
\Delta \triangleq \int_{t_0}^{t_1} f_0(x(t), u(t)) \ dt - \int_{t_0}^{t_1} f_0(x^*(t), u^*(t)) \ dt \quad (5)
\]

Now, add the null term $I_p + V^*(x_0)$ to the right-hand side of (5). In view of (1),

\[
I_p = \int_{t_0}^{t_1} \nabla V^*(x(t)) \cdot f(x(t), u(t)) \ dt \quad (6)
\]

\(^4\) That is, $\theta^1 \cap \bar{X}$ may belong to $\partial X$. 