Sufficiency Theorems for Optimal Control

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Abstract. Two sufficiency theorems, related to those of Kalman, Athans and Falb, and Boltyanskii, are derived. The derivation is based on the invariance of a certain line integral along a solution curve reaching the terminal state set. The theorems are applied to some simple problems.

1. Problem Statement

We consider a system with state equation

\[ \dot{x}(t) = f(x(t), u(t)) \]  

where \( x \) and \( f \) are \( n \)-vectors and \( u \) is an \( m \)-vector. The control \( u \) belongs to the set of admissible controls, here the set of bounded, piecewise continuous functions of \( t \) (on a bounded interval) whose values have range in a given set \( U \subset E^m \); the set \( U \) may be state-dependent. The functions \( f_i \) and \( \frac{\partial f_i}{\partial x_j} \), \( i, j = 1, 2, \ldots, n \), are continuous on \( E^n \times \bar{U} \).

The state is to be transferred from \( x(t_0) = x^0 \) to \( x(t_1) \in \theta^1 \), where \( \theta^1 \) is a given set in \( E^n \); the interval \( t_1 - t_0 \) is not specified. The performance index

\[ \int_{t_0}^{t_1} f_0(x(t), u(t)) \, dt \] 

is to be minimized, where \( f_0 \) has the same properties as \( f \). A control that accomplishes this task with respect to a given set of admissible controls is said to be optimal with respect to that set.

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3 The case of nonautonomous state equation and time-dependent end conditions can be treated in the usual manner by introducing \( x_{n+1} = t \).
2. Sufficiency Theorem 1

Here, we give a theorem that is similar to one of Kalman (Ref. 1) and of Athans and Falb (Ref. 2). Let $X$ be an open set in $E^n$. Then, we have the following theorem:

**Theorem 1.** The control $u^*(t)$ with corresponding solution $x^*(t)$, $t_0 \leq t \leq t^*_1$, is optimal with respect to all the admissible controls $u(t)$ with corresponding solutions $x(t)$, $t_0 \leq t \leq t_1$, such that $x(t_1) \in \partial I$ and $x(t) \in X \forall t \in [t_0, t_1)$, if there exists a scalar function $V^*$ of class $C^1$ on $X$ with $\lim_{x \to x^1} V^*(x) = 0$ for $x^1 \in \partial I$ and $x \in p \subset \bar{X}$, where $p$ is a solution curve of (1) generated by an admissible control, such that

\[(i) \quad f_0(x^*(t), u^*(t)) + \nabla V^*(x^*(t)) \cdot f(x^*(t), u^*(t)) = 0 \quad \forall t \in [t_0, t^*_1]\]

and

\[(ii) \quad f_0(x, u) - \nabla V^*(x) \cdot f(x, u) \geq 0 \quad \forall u \in U, \forall x \in X\]

**Proof.** Consider a solution curve $p$ of (1) from $x^0$ to $\partial I$, generated by an admissible control $u(t)$, $t_0 \leq t \leq t_1$, such that the solution $x(t) \in X \forall t \in [t_0, t_1)$. As a consequence of the hypotheses of the theorem,

\[I_p \triangleq \lim_{y \to x^1} \int_{x^0}^y \nabla V^*(x) \cdot dx = -V^*(x^0) \quad (3)\]

By (i) of the theorem, together with (1) and (3),

\[\int_{t_0}^{t^*_1} f_0(x^*(t), u^*(t)) \, dt = V^*(x^0) \quad (4)\]

Consider the difference in the value of the payoff

\[\Delta \triangleq \int_{t_0}^{t^*_1} f_0(x(t), u(t)) \, dt - \int_{t_0}^{t^*_1} f_0(x^*(t), u^*(t)) \, dt \quad (5)\]

Now, add the null term $I_p + V^*(x^0)$ to the right-hand side of (5). In view of (1),

\[I_p = \int_{t_0}^{t^*_1} \nabla V^*(x(t)) \cdot f(x(t), u(t)) \, dt \quad (6)\]

\[\text{That is, } \partial I \cap \bar{X} \text{ may belong to } \partial X.\]