Nonzero-Sum Differential Games

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Abstract. The theory of differential games is extended to the situation where there are $N$ players and where the game is nonzero-sum, i.e., the players wish to minimize different performance criteria. Dropping the usual zero-sum condition adds several interesting new features. It is no longer obvious what should be demanded of a solution, and three types of solutions are discussed: Nash equilibrium, minimax, and noninferior set of strategies. For one special case, the linear-quadratic game, all three of these solutions can be obtained by solving sets of ordinary matrix differential equations. To illustrate the differences between zero-sum and nonzero-sum games, the results are applied to a nonzero-sum version of a simple pursuit-evasion problem first considered by Ho, Bryson, and Baron (Ref. 1). Negotiated solutions are found to exist which give better results for both players than the usual saddle-point solution. To illustrate that the theory may find interesting applications in economic analysis, a problem is outlined involving the dividend policies of firms operating in an imperfectly competitive market.

1. Introduction

Since the study of differential games was initiated by Isaacs (Ref. 2) in 1954, many papers on the subject have appeared, mostly dealing with problems of the pursuit-evasion type. The differential games considered in those papers have almost always had the zero-sum property, i.e., there is a single performance criterion which one player tries to minimize and the other tries to maximize.

This paper considers a more general class of differential games, where there may be more than two players and where each player tries to minimize
his individual performance criterion. Each player controls a different set of inputs to a single system, described by a differential equation of arbitrary order. The sum of all the players’ criteria is not zero nor is it constant. Dropping the zero-sum hypothesis adds both conceptual and analytic complexity, but in the authors’ opinion it extends the utility of the theory of differential games to economic and military applications.

Very little work has been published on this subject, although Case (Ref. 3) extended some of Isaacs’ results to the nonzero-sum, N-player case for one special kind of solution. But Case did not explore the implications of dropping the zero-sum hypothesis, nor were any practical applications discussed.

Before introducing our general differential game, we illustrate some of the important conceptual differences between zero-sum games and nonzero-sum games, using simple bimatrix games of the type presented by Luce and Raiffa (Ref. 4).

In Game 1, Player 1 chooses between strategies a and b, while Player 2 simultaneously must choose x or y. The corresponding entries give the costs $J_1$, $J_2$ for the two players. For each strategy pair, $J_1 + J_2 = 0$, so the game is zero-sum. (In all games, each player wishes to minimize his own cost and is indifferent to the cost paid by the other player.) Player 2, if he is rational, always plays x, and Player 1, realizing this, plays a. This saddle-point solution is apparently the only reasonable one.

### Definition

If $J_1(s_1, ..., s_N), ..., J_N(s_1, ..., s_N)$ are cost functions for players 1, ..., N, then the strategy set \{s_1^*, ..., s_N^*\} is a Nash equilibrium strategy set if, for $i = 1, ..., N$,

$$J_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_N^*) \geq J_i(s_1^*, ..., s_N^*)$$

where $s_i$ is any admissible strategy for Player i.