Derivation and Validation of Initial-Value Methods for Boundary-Value Problems for Difference Equations

MICHAEL A. GOLBERG

Communicated by R. E. Kalaba

Abstract. In this paper, we develop the theory of invariant imbedding for general classes of two-point boundary-value problems for difference equations. In addition to deriving invariant imbedding equations, we show that the functions satisfying these equations in fact solve the original boundary-value problems.

1. Introduction

Recent work in discrete, optimal control theory (Ref. 1) has shown that many problems can be formulated as two-point boundary-value problems for difference equations. In this paper, using the theory of invariant imbedding (Ref. 2), we show how these boundary-value problems can be converted to initial-value ones.

Conversely, using methods previously developed for differential and integral equations (Refs. 3-4), we show that the solution of the initial-value problem satisfies the original boundary-value problem.

2. Boundary-Value Problem

Let $V$ be a real-vector space. Let $f(t, x)$ be a function defined on $\mathbb{Z}^+ \times V$ to $V$, where $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$. We consider the boundary-value problem

\begin{align}
    u(t + 1) &= f(t, u(t)), \quad (1) \\
    g(u(0)) + h(u(T)) &= v, \quad 0 \leq t < T, \quad T > 0, \quad (2)
\end{align}

Paper received October 13, 1970.

Lecturer, Department of Mathematics, University of Nevada, Las Vegas, Nevada.
where \( v \in V \), \((t, T) \in \mathbb{Z}^+\), and \( g \) and \( h \) map \( V \) into \( V \). By varying \( T \) between 1 and \( T^* \), and varying \( v \) in \( V \), we imbed problem (1)-(2) into a family of similar problems. We make the assumption that each boundary-value problem has a unique solution. Since the solution to the initial-value problem for (1) exists and is unique for \( t \geq 0 \), the solution of the boundary-value problem can be continued to all of \( \mathbb{Z}^+ \). This solution will be denoted by \( u(t, T, v) \) to emphasize its dependence on \( t \) and \( v \). In fact, \( u(t, T, v) \) is a function from \( \mathbb{Z}_+ \times \{1, 2, \ldots, T\} \times V \rightarrow V \). For simplicity, we will assume that \( T = \infty \).

The method of invariant imbedding seeks to replace (1)-(2) by initial-value problems. For example, if we know the final value \( u(T, T, v) \) of the solution to (1)-(2) and, in addition, if the backward Cauchy problem for (1) is uniquely solvable, then we can solve (1)-(2) by backward recursion. The problem now is to find \( u(T, T, v) \). Using invariant imbedding, we set up a difference equation for \( R(T, v) = u(T, T, v) \). Under certain conditions on \( g \) and \( h \), we can determine a complete set of initial conditions for this equation. Once this is done, solving the original boundary-value problem is reduced to solving two initial-value problems.

We will also show how the knowledge of \( R(T, v) \) can be used to give a single-sweep (Ref. 3) solution procedure for (1)-(2), analogous to the ones developed for differential equations in Refs. 3 and 5.

In the next section, we will carry out the above program under the condition that \( g + h \) has an inverse. When this condition is not satisfied, certain modifications in the previous argument need to be made. This will be done in Section 4.

3. Invariant Imbedding Equations—I

**Theorem 3.1.** (Derivation of the initial value method.) Let \( u(t, T, v) \) be defined as in Section 2. Define \( R(T, v) = u(T, T, v) \). Assume that the backward Cauchy problem for (1) is uniquely solvable. If the equation \( u(T + 1, T + 1, \bar{v}) = u(T + 1, T, v) \) can be solved uniquely for \( \bar{v} \), then the functions \( u(t, T, v) \), \( R(T, v) \) satisfy the difference equations

\[
\begin{align*}
&u(t + 1, v + h(f(T, R(T, v))) - h(R(T, v))) = u(t, T, v), \quad 0 \leq t \leq T, \\
&u(t, t, v) = R(t, v), \\
&R(T + 1, v + h(f(T, R(T, v))) - h(R(T, v))) = f(T, R(T, v)).
\end{align*}
\]

In addition, if \( g + h \) has an inverse, then \( R(T, v) \) has the initial value

\[
R(0, v) = (g + h)^{-1}(v).
\]