Implementation of Gradient Methods by Tangential Discretization$^{1,2}$

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Abstract. This paper deals with a procedure for implementing iterative methods for nonlinear programming. For constrained problems, we examine the procedure in relation to the gradient-projection method. At each iteration, the domain of suboptimization is replaced by an infinite but discrete set of points, satisfying the tangential properties for the convergence of the algorithm. It should be possible to use this procedure on other iterative methods which proceed by a series of suboptimizations, if the domain of these suboptimizations is of small dimension.

Key Words. Discretization, implementable algorithms, nonlinear constraints, gradient-projection method, mathematical programming.

1. Introduction

Most nonlinear programming methods are infinite algorithms, generating a sequence of points converging towards an optimal solution. Each iteration of the algorithms consists of solving a simpler problem, whose method of solution is assumed to be classical. Many particularly simple examples are afforded by what are known as the gradient methods: steepest ascent, conjugate gradient, variable metric, etc., which reduce the calculation to successive maximizations along half-lines. These calculations in one dimension can be performed by means of algorithms that are simple (dichotomies, for instance) but infinite. This leads to nested infinite loops of calculations, in principle a somewhat unsatisfactory situation.

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In actual practice, the inevitably limited precision of the operations makes the calculation at each iteration finite. Some authors, relatively few in number, have so far turned their attention to integrating this approximation with the theory of convergence, in order to obtain implementable methods. Without going so far as to take account of roundoff errors, these infinite algorithms implicitly require only a limited number of calculations for each iteration. However, many of these methods have the appearance of complications of classical methods, and call for new proofs of convergence.

The subject of this paper, which was motivated by the above observations, concerns a method of implementation by tangential discretization, defined below. We applied it to two examples: ascent methods in the case of maximization without constraints, and the gradient-projection method in the case of nonlinear constraints. In both cases, the partial domain, on which the suboptimization at a given iteration $k$ is carried out, is replaced by an infinite but discrete subset of points. This subset is defined by a point-to-set map of the current solution $x_k$, and satisfies the conditions of continuity and the tangential properties for convergence of the algorithm so that the proof of convergence remains unchanged. It should be possible to use this procedure on other iterative methods, which proceed by successive suboptimizations, if the subset on which the auxiliary optimization is carried out is of small dimension.

The case of ascent methods for maximization without constraints is studied in Ref. 1. Although the problem of implementing these methods was solved long ago (even before literature on the subject had stopped proposing a flood of conjugate gradient or variable metric methods, with exact maximization at each iteration), it may be interesting to consider this family of methods because of its pedagogic interest. Here, the associated discretization procedure is particularly simple and general and can provide theoretical justification for dichotomic procedures. In most of the known ascent methods, $f$ must be maximized over a half-line $D_k$ defined by the current point $x_k$ and one direction $c_k$. $D_k$ simply may be replaced by a sequence $\Delta_k$ of points of $D_k$ converging towards $x_k$, for instance, 

$$\Delta_k = \{y_i \mid y_i = x_k + c_k/2^i, i \in N\}.$$

An optimal point $x_{k+1}$ is found at the end of a finite number of trials, by calculating in succession the values $f(y_i)$, in the order $i = 0, 1, 2, \ldots$. If $f$ is a quasiconcave function, these calculations are stopped when the sequence of values $f(y_i)$ begins to decrease.

In Section 3, the gradient-projection method is studied in the case of nonlinear constraints. It is an extension of the well-known method of Rosen (Ref. 2) for maximizing a differentiable concave function on a polyhedron. It is known that this method can be extended theoretically without difficulty to