A Sufficiency Theorem for Optimal Control

G. Leitmann and H. Stalford

Abstract. A sufficiency theorem is derived for optimal control of dynamical systems governed by ordinary differential equations. A simple example is given to illustrate the application of this theorem.

1. Introduction

An important problem in optimal control theory is the determination of optimal controls and corresponding trajectories of dynamical systems. One method of finding candidates for optimal controls and trajectories is by the constructive utilization of necessary conditions for optimality such as the maximum principle (see, for example, Ref. 1). However, enabling one to assert the optimality of a candidate is often more difficult. One device for assuring optimality is the verification of sufficiency conditions.

A number of sufficiency theorems may be found in the literature. Some of these are essentially field theorems requiring the use of a sufficiently smooth function of the state variables (Refs. 2–8). Others invoke various convexity assumptions (Refs. 9–10). Recently, a rather simple and more readily verifiable sufficiency theorem was given in Refs. 11–12. It is the purpose of this investigation to extend and generalize the results of Refs. 11–12.

2. Problem Statement

Consider a system with state equation

\[ \dot{x} = f(x, v), \]  

(1)

1 Paper received February 3, 1971. This research was supported by the Office of Naval Research, Grant No. N00014-69-A-102.

2 Professor of Engineering Sciences, University of California at Berkeley, Berkeley, California.

3 Staff Member, Operations Research Group, Naval Research Laboratory, Washington, D.C.

4 For nonautonomous systems, that is, with \( f \) an explicit function of independent variable \( t \), one component of \( x \) is \( t \) itself.
where the state variable $x \in E^n$, the control variable $v \in E^m$, and the state velocity function $f$ is continuously differentiable on $E^n \times E^m$. The state space $X$ is a given subset of $E^n$. The initial state set $\theta^0$ and the terminal state set $\theta^f$ are given sets contained in $X$.

For given interval $[t_0, t_f]$, an absolutely continuous solution $\phi : [t_0, t_f] \rightarrow E^n$ of Eq. (1) for a measurable control $u : [t_0, t_f] \rightarrow E^m$ and given initial conditions is called a trajectory. A trajectory $\phi$ is said to be admissible iff $\phi(t) \in X$ for all $t \in [t_0, t_f]$, $\phi(t_0) \in \theta^0$ and $\phi(t_f) \in \theta^f$. Note that $t_0$ and $t_f$ are fixed.

Constraints on the control are considered by means of the set-valued function

$$U : X \rightarrow \text{set of all nonempty subsets of } E^m. \quad (2)$$

That is, given $x \in X$, the set $U(x)$ is the set of all the control values available at state $x$. A control $u$ is said to be admissible iff it generates at least one admissible trajectory $\phi : [t_0, t_f] \rightarrow X \cup \theta^0 \cup \theta^f$ such that $u(t) \in U(\phi(t))$ for all $t \in [t_0, t_f]$.

Let $\mathcal{J}$ denote the set of all admissible controls. We assume that $\mathcal{J}$ is nonempty. For optimality, it is required to transfer the state from $\theta^0$ to $\theta^f$, while rendering the minimum value of a given performance index

$$\int_{t_0}^{t_f} f_0(\phi(t), u(t)) \, dt, \quad (3)$$

where $f_0$ is continuously differentiable on $E^n \times E^m$, $u \in \mathcal{J}$, and $\phi$ is a corresponding admissible trajectory; that is,

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t_f} f(\phi(\tau), u(\tau)) \, d\tau. \quad (4)$$

Let $u^* \in \mathcal{J}$, and let $\phi^*$ denote an admissible trajectory generated by $u^*$. The pair $(u^*, \phi^*)$ is said to be optimal iff, for all $u \in \mathcal{J}$, with corresponding admissible trajectory $\phi$,

$$\int_{t_0}^{t_f} f_0(\phi^*(t), u^*(t)) \, dt \leq \int_{t_0}^{t_f} f_0(\phi(t), u(t)) \, dt. \quad (5)$$

---

5 If $X = E^n$, the state velocity function $f$ need not be differentiable with respect to the control variables.

6 Whereas $u$ denotes a control function, $v$ denotes its value, that is, $v = u(t)$. 

---