On Convex Vectorial Optimization in Linear Spaces\textsuperscript{1}

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Abstract. We give a new method of scalarization for convex vectorial optimization problems, with applications to best vectorial approximation and to scalar problems of optimization and best approximation.

Key Words. Vectorial and scalar optimization, convex programs, best approximation, characterization of minimal elements, existence of minimal elements, uniqueness of minimal elements.

1. Introduction

The aim of the present paper is to give a new method of scalarization for convex vectorial optimization problems, i.e., a method of obtaining the minimal elements for convex vectorial programs as solutions of suitable convex scalar programs. This approach is different from the Kuhn–Tucker scalarization and is suitable for dealing with problems of optimization with respect to a finite number of objective functions and related applications. Such applications include characterizations of elements of best vectorial approximation, simultaneous approximation of a function and its derivative, simultaneous approximation of the absolute and relative Chebyshev errors, and so on (see Refs. 1–4). In addition, we wish to mention that our approach leads to efficient computations, as suggested by Corollary 6.1 (for examples of related algorithms, see Refs. 2, 3, 5).

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In the present paper, we shall use our method of scalarization to study characterization, existence, and uniqueness of minimal elements and related problems; and we shall give some applications, in particular to best vectorial approximation and to scalar problems of optimization and best approximation. For simplicity, we shall only consider here proper convex functionals, without any further special mention.

Assume that $f$ is a proper convex functional on a real or complex linear space $E$, i.e., a functional $E \to (-\infty, +\infty]$ with $f \neq +\infty$ such that

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for all $x, y \in E$ and all $0 \leq \lambda \leq 1$. Also, assume that $G$ is a convex set in $E$. An element $g_0 \in G$ is called a solution of the convex scalar program $(G, f|_G)$ if we have

$$f(g_0) = \inf_{g \in G} f(g). \quad (1)$$

We shall denote by $\mathcal{S}_G(f)$ the set of all such elements $g_0$, i.e.,

$$\mathcal{S}_G(f) = \{ g_0 \in G \mid f(g_0) = \inf_{g \in G} f(g) \}. \quad (2)$$

In order to work with vectorial programs, we recall the usual partial ordering of the plane $R^2$: $(\alpha_1, \alpha_2) \leq (\beta_1, \beta_2)$ iff both $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. We shall use the notation

$$(\alpha_1, \alpha_2) < (\beta_1, \beta_2)$$

if

$$(\alpha_1, \alpha_2) \leq (\beta_1, \beta_2) \quad \text{and} \quad (\alpha_1, \alpha_2) \neq (\beta_1, \beta_2),$$

i.e., if both $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$, with at least one of these inequalities being strict; some authors use other notations for this case.

We recall (see, e.g., Ref. 6, Chapter 7, Section 7.4) that, if $f_1, f_2$ are two convex functionals on a linear space $E$ and if $G$ is a convex subset of $E$, an element $g_0 \in G$ is said to be a minimal element for the convex vectorial program $(G, f_1|_G, f_2|_G)$, if there exists no element $g \in G$ such that

$$(f_1(g), f_2(g)) < (f_1(g_0), f_2(g_0)). \quad (3)$$

We shall denote by $\mathcal{U}_G(f_1, f_2)$ the set of all minimal elements for the convex vectorial program $(G, f_1|_G, f_2|_G)$. It is easy to see that the set $\mathcal{U}_G(f_1, f_2)$ need not be convex, even when $G$ is a linear subspace of dimension 2.

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In Ref. 6, the term effective element is used for a dual problem of concave vectorial maximization.