On a Class of Linear Differential Games

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Dedicated to Professor A. Busemann

Abstract. Saddlepoint strategies are deduced for a class of linear, single-input differential games. Conditions necessary for a saddle-point, as well as sufficient conditions, are discussed. The results are illustrated with a simple example. For the case of one control subject to a norm bound and the other to a quadratic penalty, the results are extended to the vector case and applied to the stabilization of a system subject to norm-bounded input disturbance.

Key Words. Differential games, pursuit-evasion, stability, uncertain systems.

1. Introduction

Differential games with linear dynamics arise in a number of ways. One class of such games consists of simple games of pursuit-evasion in which the pursuer desires to minimize distance to the evader, either in the average or at a given time, whereas the evader wishes to maximize that distance; e.g., see Ref. 1. Another class arises from worst case design problems. In such problems, it is required to design a controller that stabilizes a system (plant), subject to unknown but bounded disturbances. While not a real game—in that Nature is not a sentient player—the solution to such a pseudo-game often results in a controller possessing the desired properties; e.g., see Ref. 2.

In the ensuing discussion, we treat a class of differential games with linear dynamics, single input, and quadratic cost. We consider two cases, one in which both players' controls are norm-bounded, and the
other in which only one player's control is so bounded and the other's is subject to a quadratic penalty.

2. System with Norm-Bounded Controls

Consider the dynamical system

\[
\dot{x}(t) = A(t) x(t) + b(t) u(t) + c(t) v(t),
\]

\[
x(t_0) = x_0, \quad t \in [t_0, t_1],
\]

where \( x(t) \in \mathbb{R}^n \); \( A(\cdot) \) is an \( n \times n \) matrix function, continuous on \( \mathbb{R}^1 \); \( b(\cdot) \) and \( c(\cdot) \) are \( n \times 1 \) vector functions, continuous on \( \mathbb{R}^1 \); and \( u(\cdot) \) and \( v(\cdot) \) are scalar-valued functions, Lebesgue measurable on \( \mathbb{R}^1 \) satisfying

\[
u(t) \leq v, \quad v = (v: |v| \leq \rho),
\]

With this system, we associate a cost

\[
J = \int_{t_0}^{t_1} x'(t) Q(t) x(t) dt,
\]

where \( Q(\cdot) \) is an \( n \times n \) symmetric matrix function, continuous on \( \mathbb{R}^1 \).

For system (1)-(3), we seek a saddlepoint strategy pair \( \{p^*(\cdot), e^*(\cdot)\} \) in the class of strategy pairs \( \{p(\cdot), e(\cdot)\} \) satisfying the hypotheses of Ref. 3, e.g., Borel measurable from \( \mathbb{R}^n \times (-\infty, t_1] \to \mathbb{R}^1 \), such that

\[
u(t) = p(x(t), t), \quad \varphi(t) = e(x(t), t).
\]

Here, \( p(\cdot) \) is the minimizer's strategy and \( e(\cdot) \) that of the maximizer.

The necessary conditions for a saddlepoint\(^4\) allow one to state certain conditions to be met by a saddlepoint candidate \( \{p^*(\cdot), e^*(\cdot)\} \). For all \( (x, t) \in \mathbb{R}^n \times (-\infty, t_1) \), let\(^5\)

\[
D_1 = \{(x, t): b'Px + b'R(-b\rho_u + c\rho_v) > 0 \text{ and } c'Px + c'R(-b\rho_u + c\rho_v) > 0\},
\]

\[
D_2 = \{(x, t): b'Px + b'R(-b\rho_u - c\rho_v) > 0 \text{ and } c'Px + c'R(-b\rho_u - c\rho_v) < 0\},
\]

\[
D_3 = \{(x, t): b'Px + b'R(b\rho_u + c\rho_v) < 0 \text{ and } c'Px + c'R(b\rho_u + c\rho_v) > 0\},
\]

\[
D_4 = \{(x, t): b'Px + b'R(b\rho_u - c\rho_v) < 0 \text{ and } c'Px + c'R(b\rho_u - c\rho_v) < 0\},
\]

\[
D_5 = \left\{(x, t): (x, t) \in \bigcup_{i=1}^{4} D_i \right\}.
\]

\(^4\) For application of the necessary condition (e.g., Refs. 3–4), \( A(\cdot), b(\cdot), c(\cdot), \) and \( Q(\cdot) \) are restricted to \( C^1 \) functions.

\(^5\) \( (\cdot)^c \) denotes the complement of \( (\cdot) \) in \( \mathbb{R}^n \times (-\infty, t_1) \).