A Second-Order Lagrangian Condition for Restricted Control Problems

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Abstract. We derive a second-order condition for optimal control problems defined by ordinary differential equations with time-varying convex restrictions on controls and with endpoint constraints. This condition generalizes the accessory minimum problem of the calculus of variations.

Key Words. Control theory, second-order conditions, minimizing controls, accessory minimum problem.

1. Lagrangian Condition

The purpose of this paper is to derive a second-order condition that must be satisfied by minimizing controls in problems defined by ordinary differential equations with restrictions on the controls and the trajectory endpoint. We refer to this condition as Lagrangian, because it is derived using Lagrangian (or weak) variations as distinguished from Weierstrassian (or strong) variations and from variations in the space of relaxed controls. This Lagrangian condition generalizes a classical necessary condition of the calculus of variations referred to as the accessory minimum problem (Ref. 1, p. 228). A different second-order condition, based on variations in the space of relaxed controls, is derived in Ref. 2, where the reader can also find references to other second-order conditions in the literature as well as some illustrative examples.

We consider the following problem of optimal control: Minimize $h^0(x(t_1))$ over the class of admissible couples $(x, u)$, which we define as the class of couples $(x, u)$ such that

$u: [t_0, t_1] \rightarrow U \subset \mathbb{R}^k$

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is (Lebesgue) measurable,

\[ x: [t_0, t_1] \to V \subset \mathbb{R}^n \]

is absolutely continuous, and \((x, u)\) satisfies the relations

\[ \begin{align*}
\dot{x}(t) &= f(t, x(t), u(t)) \quad \text{a.e. in } T = [t_0, t_1], \\
x(t_0) &= 0, \quad h^1(x(t_1)) = 0, \\
u(t) &\in U^w(t) \quad \text{a.e. in } T.
\end{align*} \tag{1} \]

We assume that \(V\) is an open subset of \(\mathbb{R}^n\) and

\[ f: T \times V \times U \to \mathbb{R}^n, \quad h^0: V \to \mathbb{R}, \quad h^1: V \to \mathbb{R}^m \]

are such that, for all \((t, v, u) \in T \times V \times U,\)

\[ f(\cdot, v, u) \text{ is measurable and } f(t, \cdot, \cdot), h^0, \text{ and } h^1 \text{ are twice continuously differentiable with uniformly bounded derivatives.} \]

We also postulate that \(U\) is a bounded open subset of \(\mathbb{R}^k\) and the sets \(U^w(t), t \in T,\) are closed and convex subsets of some compact \(U_0 \subset U\) and such that

\[ \{t \in T| U^w(t) \cap G \neq \emptyset\} \]

is measurable for every open \(G \subset \mathbb{R}^k.\)

We refer to a couple \((\bar{x}, \bar{u})\) as extremal if it satisfies Pontryagin’s minimum principle as generalized to this problem (Ref. 2, Section VI.2.3, pp. 357 ff), i.e., if \((\bar{x}, \bar{u})\) is admissible and there exists

\[ l = (l_0, l_1) \in \mathbb{R} \times \mathbb{R}^m \]

such that

\[ l \neq 0, \quad l_0 \in \{0, 1\}, \]

and

\[ z(t)^T f(t, \bar{x}(t), \bar{u}(t)) = \min_{u \in U^w(t)} z(t)^T f(t, \bar{x}(t), u) \quad \text{a.e. in } T, \]

where \(z,\) the dual function, is defined by

\[ z(t)^T = \sum_{i=0}^{1} l_i^T h_i^0(\bar{x}(t_1)) Z(t), \]

\(Z\) is the solution of the matrix-differential equation

\[ Z(t) = I + \int_{t}^{t_1} Z(\tau) f_o(\tau, \bar{x}(\tau), \bar{u}(\tau)) d\tau, \quad t \in T, \]