A General Sufficiency Theorem for Minimax Control

W. Schmitendorf

Communicated by G. Leitmann

Abstract. This paper considers optimal control problems where there is uncertainty in the differential equations describing the system. A minimax optimality criterion is used, and sufficient conditions for a control to be a minimax control are presented. These conditions are more general than those given in Refs. 1 and 2.

Key Words. Minimax problems, sufficient conditions.

1. Introduction

Sufficient conditions for a control to be a minimax control have been presented in Refs. 1–2. Here, we present a more general sufficiency result, Theorem 3.1, and also show that the sufficient conditions of Refs. 1–2 are just special cases of this new set of conditions. Theorem 3.1 applies to problems with more general terminal conditions and cost functions than does Ref. 2 and can be used to verify that a control is a minimax control for some problem where the results of Refs. 1–2 fail to yield any information.

2. Problem Formulation

Consider the differential system

\[ \dot{x}(t) = f(x(t), u(t), v(t)), \]

where the state \( x(t) \in \mathbb{R}^n \), the control \( u(t) \in \mathbb{R}^{m_1} \), and the disturbance \( v(t) \in \mathbb{R}^{m_2} \). We assume that \( f(\cdot, \cdot, \cdot) \) is \( C^1 \) on \( \mathbb{R}^n \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \). The playing

---

1 This research was supported by AFOSR under Grant No. 76-2923.
2 Professor, Mechanical Engineering Department, Northwestern University, Evanston, Illinois.

463
space is denoted by $X \times T$. $X$ is an open set in $\mathbb{R}^n$, and $T = [t_0, t_f]$ where $t_0$ and $t_f$ are specified. At $t_0$, the initial state $x_0$ is specified. At the final time, we require $x(t_f) \in \theta$, where the target set $\theta$ is a given set in the closure of $X$.

Let $\mathcal{U}$ denote the set of piecewise continuous functions from $T$ into $\mathbb{R}^{m_1}$, and let $U$ be a given subset of $\mathbb{R}^{m_1}$. Define

$$
\mathcal{M}_1 = \{u(\cdot) : u(\cdot) \in \mathcal{U} \text{ and } u(t) \in U, t \in T\}.
$$

Let $V$ be a given subset of $\mathbb{R}^{m_2}$, and let $\mathcal{V}$ denote the set of piecewise continuous functions $q(\cdot, \cdot) : X \times T \rightarrow V$ with respect to some decomposition of $X \times T$.

$$
\mathcal{M}_2 = \{q(\cdot, \cdot) : q(\cdot, \cdot) \in \mathcal{V} \text{ and } q(x, t) \in V \text{ for all } (x, t) \in X \times T\}.
$$

For a given pair $[u(\cdot), q(\cdot, \cdot)]$, $u(\cdot) \in \mathcal{M}_1$ and $q(\cdot, \cdot) \in \mathcal{M}_2$, a solution of (1) from $(x_0, t_0)$ will be denoted by $x(\cdot)$ and called a trajectory. A terminating trajectory is a trajectory satisfying $x(t) \in X$ for all $t \in [t_0, t_f]$ and $x(t_f) \in \theta$. A pair $[u(\cdot), q(\cdot, \cdot)]$ is playable at $(x_0, t_0)$ if it generates a terminating trajectory. The pair $[u(\cdot), q(\cdot, \cdot)]$ may generate more than one terminating trajectory from $(x_0, t_0)$, and $\Phi(u(\cdot), q(\cdot, \cdot))$ will denote the set of all such trajectories.

For $u(\cdot) \in \mathcal{M}_1$, $\mathcal{N}(u(\cdot))$ is the set of all $q(\cdot, \cdot) \in \mathcal{M}_2$ such that $[u(\cdot), q(\cdot, \cdot)]$ is playable at $(x_0, t_0)$. We shall say that $u(\cdot)$ is admissible if $u(\cdot) \in \mathcal{M}_1$ and $\mathcal{N}(u(\cdot)) \neq \emptyset$.

For playable pair $[u(\cdot), q(\cdot, \cdot)]$ and trajectory $x(\cdot) \in \Phi(u(\cdot), q(\cdot, \cdot))$, the cost is defined by

$$
J(u(\cdot), q(\cdot, \cdot), x(\cdot)) = g(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), q(x(t), t)) \, dt.
$$

For an admissible $u(\cdot)$, $(q(\cdot, \cdot), x(\cdot)) \in \mathcal{N}(u(\cdot))$ iff $q(\cdot, \cdot) \in \mathcal{N}(u(\cdot))$ and $x(\cdot) \in \Phi(u(\cdot), q(\cdot, \cdot))$. We now define a minimax control.

**Definition 2.1.** Let $u^*(\cdot)$ be admissible. Then, $u^*(\cdot)$ is a minimax control iff

$$
\sup_{(q(\cdot, \cdot), x(\cdot)) \in \mathcal{N}(u^*(\cdot))} J(u^*(\cdot), q(\cdot, \cdot), x(\cdot)) \leq \sup_{(q(\cdot, \cdot), x(\cdot)) \in \mathcal{N}(u(\cdot))} J(u(\cdot), q(\cdot, \cdot), x(\cdot))
$$

for all admissible $u(\cdot)$.

---

For example, see page 42 of Ref. 3.