Higher-Order Necessary Conditions in Abstract Mathematical Programming$^{1,2}$

K. H. HOFFMANN$^3$ AND H. J. KORNSTAEDE$^4$

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Abstract. We prove necessary extremum conditions for general nonlinear optimization problems in ordered topological vector spaces. For that reason, we define variational derivatives of higher order and introduce proper variations. Especially assuming certain weak hypotheses, we establish maximum principles of higher order.

Key Words. Optimization theory, variational sets, variational derivatives, necessary conditions, maximum principles.

1. Introduction

Let $X, Y_f, Y_g, Y_h$ be real separated topological vector spaces, and let $Y_f$ and $Y_h$ be endowed with an order structure induced by proper convex cones $C_f \subset Y_f$ and $C_h \subset Y_h$, respectively, which have nonempty topological cores $\mathcal{C}_f \neq \emptyset$ and $\mathcal{C}_h \neq \emptyset$, respectively. We follow general conventions and mean by $y \leq \theta$ that $y \in C_f$ and by $y < \theta$ that $y \in C_f^\circ$. The topological dual of a space $Y$ is denoted by $Y^*$, and the algebraic dual by $Y'$. By $C^*$, we mean the polar set of the cone $C$. Let $D \subset X$ be a nonempty subset, and let

$$f : D \to Y_f, \quad g : D \to Y_g, \quad h : D \to Y_h$$

be fixed mappings.

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$^3$Professor, III. Mathematisches Institut, Freie Universität Berlin, Berlin, Germany.

$^4$Assistenzprofessor, Fachbereich Mathematik, Technische Universität Berlin, Berlin, Germany.
We consider the following nonlinear optimization problem. Minimize \( f \), in the sense of the order explained in \( Y_f \) by \( C_f \), with respect to the explicit side conditions \( g(x) = 0 \) and \( h(x) \leq \theta \).

\( x_0 \in M \) is called a (global) minimal element of \( M \) with respect to \( f \), if

\[
F[x_0] \cap M = \emptyset,
\]
where

\[
F[x_0] := \{ x \in D \mid f(x) < f(x_0) \},
\]

\[
G := \{ x \in D \mid g(x) = 0 \},
\]

\[
H := \{ x \in D \mid h(x) \leq \theta \},
\]

\[
M := G \cap H.
\]

On the same line, an element \( x_0 \in M \) is called a local minimal element of \( M \) with respect to \( f \), if there is a neighborhood \( U \) of \( x_0 \) such that

\[
F[x_0] \cap M \cap U = \emptyset.
\]

The aim of this paper is to develop necessary extremum conditions for global and local minimal elements of \( M \) with respect to \( f \). Without any restriction and toward a unified presentation of the theory, we may assume that

\[
f(x_0) = \theta
\]
and define

\[
F := \{ x \in D \mid f(x) < \theta \}.
\]

Because we are interested in necessary conditions of higher order \((m \geq 1)\), we extend the concept of tangent cone, originally introduced by Bouligand (Ref. 1), Dubovitskii and Milyutin (Refs. 2, 3), Hestenes (Ref. 4), and others have established tangent cones with great success in optimization theory. For elements \( x_0, x_1, \ldots, x_{m-1} \in X \), we define variational sets of order \( m \), \( m \in \mathbb{N}_0 \), of a subset \( A \subseteq X \) denoted by

\[
K^{(m)}[A] := K[A; x_0, \ldots, x_{m-1}] \quad \text{and} \quad K^{(m)}(A) := K(A; x_0, \ldots, x_{m-1}),
\]
respectively. In the special case \( m = 1 \), these variational sets coincide with the cones considered by Dubovitskii-Milyutin. Moreover, these authors point out what one should do in the case \( m = 2 \). Our considerations start with the easily proved necessary extremum condition

\[
K^{(m)}[F \cap M] = \emptyset.
\]