On the Numerical Determination of Optimal Inputs

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Abstract. The design of optimal inputs for linear and nonlinear system identification involves the maximization of a quadratic performance index subject to an input energy constraint. In the classical approach, a Lagrange multiplier is introduced whose value is an unknown constant. In recent papers, the Lagrange multiplier has been determined by plotting a curve of the Lagrange multiplier as a function of the critical interval length or a curve of input energy versus the interval length. A new approach is presented in this paper in which the Lagrange multiplier is introduced as a state variable and evaluated simultaneously with the optimal input. Numerical results are given for both a linear and a nonlinear dynamic system.

Key Words. Optimal inputs, system identification, Lagrange multipliers, nonlinear dynamic systems, Newton–Raphson method.

1. Introduction

The estimation accuracy for dynamic system identification is maximized by the use of optimal inputs. The design of optimal inputs for linear system identification (Refs. 1–5) and nonlinear system identification (Refs. 6–9) has been the subject of several recent papers. However, the numerical determination of the optimal inputs is far from trivial. The performance index for the optimal input is selected such that the sensitivity of the measured state variables to the unknown parameters is maximized subject to an input energy constraint. The performance index is maximized using Pontryagin’s maximum principle. The solution requires the evaluation of a two-point boundary-value problem.

Using a quadratic performance criterion, the optimal input is determined such that the integral

\[ M = \int_0^T x_2^2(t) \, dt \quad (1) \]

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is maximized, subject to the input energy constraint

\[ E = \int_0^T y^2(t) \, dt, \quad (2) \]

where

\[ x_a(t) = \frac{\partial x(t)}{\partial a}; \quad (3) \]

here, \( x(t) \) is the state variable, \( y(t) \) is the input, and \( a \) is an unknown system parameter. The performance index is maximized via the classical method by the maximization of the integral

\[ J = \max_y \frac{1}{2} \int_0^T \left[ x_a^2(t) - q(y^2(t) - E/T) \right] \, dt, \quad (4) \]

where \( q \) is the Lagrange multiplier and is equal to a constant. The magnitude of the Lagrange multiplier must be selected such that the input energy constraint is satisfied.

In Ref. 3, Mehra shows that, for a linear system with homogeneous boundary conditions, the solution exists only for certain values of \( q \) which are the eigenvalues of the two-point boundary-value problem. The eigenvalues \( q \) are functions of the interval length \( T \). For a fixed \( q \), the critical length \( T \) can be determined by the integration of the Riccati-matrix equation. When the elements of the Riccati-matrix equation become very large, the critical length has been reached. By integrating for several values of \( q \), a curve relating \( q \) to \( T \) can be obtained.

Kalaba and Spingarn show in Ref. 5 that, for a linear system with nonhomogeneous boundary conditions, the performance index increases as the critical length is approached for a given value of \( q \). The desired input energy is obtained by plotting a curve of input energy versus interval length \( T \) and finding the \( T \) corresponding to the desired input energy. The optimal input for a system with homogeneous boundary conditions is nearly identical to that of the system with nonhomogenous boundary conditions when the input energy is the same and the latter has a small initial condition with terminal time near the critical length.

For both homogeneous and nonhomogeneous boundary conditions, the Lagrange multiplier \( q \) must be found by trial-and-error such that the input energy constraint is satisfied. In this paper, a new approach for the numerical determination of optimal inputs is presented. The Lagrange multiplier \( q \) is introduced as a state variable. The solution simultaneously yields the optimal input and the value of \( q \) for which the input energy constraint is satisfied. The new approach is based on the method of solution of the isoperimetric problem presented in Ref. 10. The method is applicable to both linear and nonlinear systems.