A System of Inequalities and Nondifferentiable Mathematical Programming

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Abstract. A system of inequalities involving nondifferentiable functions, originally introduced by Eisenberg, is extended. This result is then applied to establish optimality criteria and a dual theorem for a nondifferentiable mathematical program involving quasiconvexity in its equality constraints and inequality constraints. These are natural extensions of results recently established by Mond.

Key Words. Positive-semidefinite symmetric matrices, optimality, duality, quasiconvexity.

1. Introduction

The purpose of this paper is to generalize or extend results of Eisenberg (Ref. 1) and Mond (Ref. 2). For a given vector \( \alpha \in \mathbb{R}^n \), a real \( m \times n \) matrix \( A \) and an \( n \times n \) real symmetric positive semidefinite matrix \( C \), Eisenberg defined

\[
K = \mathbb{R}^n \cap \{x : Ax \leq 0\}
\]

and

\[
f_0(x) = \alpha^T x + (x^T C x)^{1/2}
\]

for \( x \in \mathbb{R}^n \) and showed that

\[
f_0(x) \geq 0
\]

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3 To avoid confusion, we denote Eisenberg's \( f \) by \( f_0 \).
for all $x \in K$ iff there exist $z \in \mathbb{R}^n$, $\pi \in \mathbb{R}^m$ such that

$$A z \leq 0, \quad \pi \geq 0, \quad \pi^t A + \alpha^t + z^t C = 0, \quad z^t C z \leq 1.$$ 

Mond (Ref. 2) applied Eisenberg's result to the following nonlinear programming problem:

$$\text{minimize } F(x) = f(x) + (x^t B x)^{1/2}, \quad \text{subject to } g(x) \leq 0,$$

where $f$ and $g$ are differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^m$, respectively, and $B$ is an $n \times n$ symmetric positive semidefinite matrix. He established a necessary optimality condition of Kuhn–Tucker type under a constraint qualification which was later shown by Mond and Schechter (Ref. 3) to be implied by the generalized Slater condition. Mond (Ref. 2) also proved a sufficient optimality theorem assuming that $f$ is convex and $g$ is concave. Finally, he defined the dual problem and established Wolfe's (Ref. 4) duality theorems and converse duality theorems under this set-up.

To extend Eisenberg's result, we define

$$K_0 = \mathbb{R}^n \cap \{x : Ax \leq 0, D x = 0\},$$

$$f_0(x) = \alpha^t x + (x^t C x)^{1/2},$$

where $\alpha$ and $A$ are same as before, $D$ is a $k \times n$ matrix, and $C$ is an $n \times n$ symmetric positive semidefinite matrix. Under this set-up, in Section 2, we extend Eisenberg's Lemma 2 (Ref. 1) by introducing equalities via the matrix $D$.

In Section 3, we generalize Mond's (Ref. 2) optimality criteria by considering the following primal programming problem [Problem (P)]:

$$\text{minimize } F(x) = f(x) + (x^t C x)^{1/2}, \quad x \in K^*$$

where

$$K^* = \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0\},$$

and where $f$ and $g$ are the same as in Mond (Ref. 2) and $h$ is a $k$-dimensional vector-valued function. For sufficiency, we relax Mond's convexity restriction on $-g$ to quasiconvexity and require $h$ to be quasiconvex.

In Section 4, we define the dual problem and extend the Hanson–Hurad strict converse duality theorem by requiring $h$ to be quasiconvex and not both quasiconvex and quasiconcave as in Ref. 4.

### 2. A System of Inequalities

Let $K_0$, $A$, $D$, $C$ be as in Section 1. Let $\beta$ be a fixed vector in $\mathbb{R}^n$. Then, we have the following theorem.