Duality Theory in Multiobjective Programming

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Abstract. In this paper, a multiobjective programming problem is considered as that of finding the set of all nondominated solutions with respect to the given domination cone. Two point-to-set maps, the primal map and the dual map, and the vector-valued Lagrangian function are defined, corresponding to the case of a scalar optimization problem. The Lagrange multiplier theorem, the saddle-point theorem, and the duality theorem are derived by using the properties of these maps under adequate convexity assumptions and regularity conditions.

Key Words. Domination structures, cone extreme points, primal map, vector-valued Lagrangian function, dual map.

1. Introduction

In recent years, the analysis of a programming problem with several objectives conflicting with one another has been a focal issue. Such a multiobjective optimization problem reflects the complexity of the real world and is encountered in various fields. An optimal solution to such a problem is ordinarily chosen from the set of all Pareto-optimal solutions (noninferior solutions) to it. Many authors have developed the necessary and/or sufficient conditions for Pareto optimality (Refs. 1–4). In Ref. 5, Yu proposed the concept of the domination structure defined by a convex cone and provided the procedure for obtaining the nondominated solutions. These results give a more general concept of solutions.

On the other hand, the duality theory (Lagrangian saddle-point result) has been another focal issue for a long time, especially in convex programming. The conditions for Pareto optimality in a multiobjective problem are some extensions of those results.

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In this paper, we consider a multiobjective programming problem as a problem which finds the set of solutions instead of a single solution. Here, we mean by solutions to the problem the cone extreme points of a given feasible set with respect to a given convex cone which specifies the domination structure of the decision-maker.

We will define the primal map and the dual map for a multiobjective programming problem under the assumption that the functions and the set defining the problem are convex with respect to the cone specifying the domination structure. And we will develop the discussion in parallel with the case of a scalar convex programming problem. Our development closely follows that of Luenberger (Ref. 6).

Before we go further, for convenience let us introduce the following notations. Let

\[ x = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad y = (y_1, y_2, \ldots, y_n) \]

be vectors in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Then,

(i) \( x < y \) iff \( x_i < y_i \) for all \( i = 1, \ldots, n \);

(ii) \( x \leq y \) iff \( x_i \leq y_i \) for all \( i = 1, \ldots, n \).

The zero vector \((0, 0, \ldots, 0)\) in \( \mathbb{R}^n \) is also denoted by \( 0 \). The nonnegative orthant in \( \mathbb{R}^n \) is denoted by

\[ \mathbb{R}_+^n = \{ x \in \mathbb{R}^n \mid x \geq 0 \} \]

Given a set \( A \subset \mathbb{R}^n \), its closure and interior will be denoted by \( \bar{A} \) and \( \text{int} A \), respectively. Given two sets \( A \) and \( B \), their addition is defined by

\[ A + B = \{ a + b \mid a \in A, \ b \in B \} \]

also, given a scalar \( k \), the scalar multiplication of the set \( A \) by \( k \) is defined by

\[ kA = \{ ka \mid a \in A \} \]

If \( A \) contains only a single point \( \bar{a} \), its sum with the set \( B \) is simply denoted by \( \bar{a} + B \).

\[ \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \]

will denote the inner product of the two vectors \( x \) and \( y \). For an arbitrary set \( D \) in \( \mathbb{R}^n \), we define its (positive) polar cone by

\[ D^* = \{ v \in \mathbb{R}^n \mid \langle v, d \rangle \geq 0 \ \text{for all} \ d \in D \} \]

Finally, given a set \( A \), the cone generated by \( A \) is defined and denoted by

\[ [A] = \{ b \mid b = ka, \ a \in A, \ k \in \mathbb{R}_+^1 \} \].