A Dynamic Programming Approach to the Maximum Principle of Distributed-Parameter Systems

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Abstract. This paper provides a dynamic programming approach to the maximum principle for the optimal control of systems with distributed parameters. The process of the systems under consideration is governed by a partial differential equation.

Key Words. Dynamic programming, maximum principle, optimal control, distributed-parameter systems, Frechet derivatives.

1. Introduction

Problems in the optimal control of systems with distributed parameters were formulated by Butkovskii and Lerner in Ref. 1. In Ref. 2, Sirazetdinov considered processes governed by a quasilinear partial differential equation and established the maximum principle for these processes. In the following, we extend the idea used in Ref. 3 to study, by the method of dynamic programming, optimal control problems with systems of distributed parameters and generalize the maximum principle obtained in Ref. 2 to processes governed by a partial differential equation which is not necessarily quasilinear. We first study the case in which both time and state variables are discrete. This gives us some geometric understanding of the quantity $p$, called an impulse or a distributed impulse. Secondly, we consider the case in which both time and state are continuous, and obtain the maximum principle in this case. Finally, we study a special case in which the control depends only on the time.

Throughout this paper, we focus our attention on new approaches and concepts and not on rigorous arguments justifying the existence of solutions

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of the partial differential equations under consideration and the existence of the optimal control. We assume that the optimal value function, which will be defined later, is differentiable. For simplicity and easy understanding of our new derivations and notations, we also assume that both state variable and control variable are real-valued functions. The method works for any n-dimensional state function and m-dimensional control function.

2. Discrete-Time, Discrete-State Version

Let us consider the problem of finding the control u, which is defined on \{(j\Delta t, i\Delta x)\}|j = 0, ..., N - 1; i = 1, ..., M\} and takes the values \(u(j\Delta t, \Delta x), \ldots, u(j\Delta t, M\Delta x)\) at stage j for \(j = 0, \ldots, N - 1\), such that \(u\) minimizes

\[
\sum_{j=0}^{N-1} \sum_{i=1}^{M} \left[ g(j\Delta t, i\Delta x, v(j\Delta t, i\Delta x), \frac{v(j\Delta t, i\Delta x) - v(j\Delta t, (i-1)\Delta x)}{\Delta x}, u(j\Delta t, i\Delta x)\Delta x \Delta t) \right],
\]

where \(v\) is a function on \{(j\Delta t, i\Delta x)\}|j = 0, ..., N; i = 0, ..., M\} and is governed by the difference equations

\[
v((j+1)\Delta t, i\Delta x) = v(j\Delta t, i\Delta x) + f(j\Delta t, i\Delta x, v(j\Delta t, i\Delta x), \frac{v(j\Delta t, i\Delta x) - v(j\Delta t, (i-1)\Delta x)}{\Delta x}, u(j\Delta t, i\Delta x)\Delta t)
\]

for \(i = 1, \ldots, M\) and \(j = 0, \ldots, N - 1\), with initial conditions

\[
v(0, i\Delta x) = v^0(0, i\Delta x)
\]

for \(i = 1, \ldots, M\), and boundary conditions

\[
v(j\Delta t, 0) = v^0(j\Delta t, 0)
\]

for \(j = 0, \ldots, N - 1\). We assume that \(g\) and \(f\) are continuously differentiable with respect to their arguments. We call \(\Delta x\) the duration of a state and \(X = M\Delta x\) the final state. Similarly, \(\Delta t\) is the duration of a state and \(T = N\Delta t\) is the final time.

For convention, we use the notations \(u_{ij}, v_{ij}\) for \(u(j\Delta t, i\Delta x), v(j\Delta t, i\Delta x)\), respectively, and use \(v_{xij}\) for \([v(j\Delta t, i\Delta x) - v(j\Delta t, (i-1)\Delta x)]/\Delta x\), since the last term approaches \(v_x(j\Delta t, i\Delta x)\) as \(\Delta x \to 0\). Using these notations, the problem can be restated as follows.

**Problem D.** Minimize

\[
\sum_{j=0}^{N-1} \sum_{i=1}^{M} g(j, i, v_{ij}, v_{xij}, u_{ij})\Delta x \Delta t,
\]