TECHNICAL NOTE

An Application of Guignard’s Generalized Kuhn–Tucker Conditions

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Abstract. Necessary conditions are given for a real-valued function to have a minimum subject to an inequality constraint. Under the appropriate hypotheses, the problem is demonstrated to be a special case of the type of problem to which Guignard’s Kuhn–Tucker theorem can be applied.

Key Words. Banach spaces, Fréchet derivatives, convex cones, tangent cones.

1. Introduction

The purpose of this note is to present a multiplier rule for minimizing a differentiable function subject to an inequality constraint in which the cone defining the constraint is not required to have a nonempty interior.

Throughout the paper, it is assumed that $X$ and $Y$ are Banach spaces, $f: X \rightarrow \mathbb{R}$, $g: X \rightarrow Y$, $x_0 \in X$, and $B$ is a closed convex cone in $Y$. Further, it is assumed that $f$ has a Fréchet derivative $Df(x_0)$ at $x_0$ and $g$ has a Fréchet derivative which is continuous on some neighborhood of $x_0$ and for which $Dg(x_0)$ is onto $Y$.

2. Preliminary Definitions and Theorems

For completeness, a definition and three known results are stated. Definition 2.1 and Theorem 2.1 are both due to Varaiya (Ref. 3).
Definition 2.1. Let \( A \subseteq X \) and \( x_0 \in A \). The tangent cone of \( A \) at \( x_0 \), denoted \( TC[A; x_0] \), is the set of all elements \( k \) of \( X \) such that there exist sequences \( (x_n) \) of elements of \( A \) and \( (\lambda_n) \) of nonnegative reals for which
\[
x_n \to x_0 \quad \text{and} \quad \lambda_n(x_n - x_0) \to k.
\]

Theorem 2.1. Let \( A \subseteq X \). If \( f \) has a min on \( A \) at \( x_0 \), then
\[
TC[A; x_0] \subseteq Df(x_0)^+,
\]
where
\[
Df(x_0)^+ = \{ h \in X \mid Df(x_0)h \geq 0 \}.
\]

The next theorem is due to Flett (Ref. 1, Lemma 1).

Theorem 2.2. For \( q \in X \), define the set \( L \) by
\[
L = \{ \lambda Dg(x_0)q \mid \lambda \geq 0 \}.
\]
Then,
\[
q \in TC[g^{-1}(L + g(x_0)); x_0].
\]

The final result to be stated is due to Guignard (Ref. 2, Theorem 2). The following notation is used. For a subset \( S \) of a Banach space \( Z \), let
\[
S^+ = \{ z^* \in Z^* \mid z^*s \geq 0 \text{ for all } s \in S \},
\]
where \( Z^* \) denotes the topological dual of \( Z \), and let \( \overline{co}S \) denote the closed convex hull of \( S \).

Let \( H \) denote the subset of \( X^* \) consisting of all linear functionals \( x^* \) for which there exists
\[
y^* \in (\overline{co} TC[B; g(x_0)])^+,
\]
with
\[
x^* = y^*Dg(x_0).
\]
Let \( C \) denote a subset of \( X \), and let
\[
A = C \cap g^{-1}(B).
\]
Let
\[
K = Dg(x_0)^{-1}(\overline{co} TC[B; g(x_0)]).
\]
The necessary conditions of Guignard's theorem can now be stated.