Fundamental Matrix
and Two-Point Boundary-Value Problems

S. M. Roberts¹ and J. S. Shipman²

Abstract. Theoretically, the solution of all linear ordinary differential equation problems, whether initial-value or two-point boundary-value problems, can be expressed in terms of the fundamental matrix. The examination of well-known two-point boundary-value methods discloses, however, the absence of the fundamental matrix in the development of the techniques and in their applications. This paper reveals that the fundamental matrix is indeed present in these techniques, although its presence is latent and appears in various guises.


1. Introduction

A variety of methods have been proposed to solve two-point boundary-value problems for systems of ordinary differential equations (Refs. 1–5). While many of these techniques seem different, they do have several common themes. First, they all involve integrating a system of ordinary differential equations related to the original system and then solving a system of linear algebraic equations to find the missing initial conditions or corrections to the missing initial conditions. A second common theme is that all of the methods are realizations of Newton’s method when considered in the appropriate solution space. A third theme, which we dwell on here and which is usually unmentioned or ignored, is that all the methods for systems of linear ordinary differential equations have a common root in the representation of the solution in terms of the fundamental matrix.

In this paper, we describe how the fundamental matrix appears in the development of four two-point boundary-value techniques, the method of adjoints (Refs. 1 and 3), the method of complementary functions (Refs. 1

¹Advisory Analyst, IBM Scientific Center, Palo Alto, California.
²Advisory Analyst, IBM, Federal Systems Division, Gaithersburg, Maryland.
and 3), Miele's method of particular solutions (Ref. 2), and Scott's method of invariant imbedding (Refs. 4 and 5).

2. Fundamental Matrix

Consider the two-point boundary-value problem with the system of \( n \) inhomogeneous linear ordinary differential equations

\[
y'(t) = A(t)y(t) + f(t), \quad t_0 \leq t \leq t_f, \tag{1}
\]

where \( y(t) \) and \( f(t) \) are \( n \)-vectors and \( A(t) \) is an \( n \times n \) matrix and with the explicit boundary conditions

\[
y_i(t_0) = c_i, \quad i = 1, 2, \ldots, r, \tag{2}
\]
\[
y_{i_m}(t_f) = c_{i_m}, \quad m = 1, 2, \ldots, n - r.
\]

The solution to (1) can be expressed in terms of its fundamental matrix

\[
y(t) = M(t)y(t_0) + \int_{t_0}^{t} M(t)M^{-1}(\theta)f(\theta) \, d\theta, \tag{3}
\]

where the fundamental matrix \( M(t) \), an \( n \times n \) matrix, satisfies the initial-value problem

\[
M'(t) = A(t)M(t), \tag{4}
\]
\[
M(t_0) = I, \tag{5}
\]

with \( I \) the \( n \times n \) identity matrix.

It is possible theoretically to solve (1)–(2) directly by (3). This requires integrating (4) to obtain \( M(t) \) and \( M(t)^{-1} \) over the interval \([t_0, t_f]\). A system of \( n - r \) equations for (3) is evaluated at \( t = t_f \) and solved for the \( n - r \) missing initial conditions. Upon rearranging (3), the system for the missing initial conditions appears as

\[
\begin{bmatrix}
    m_{1,r+1}(t_f) & m_{1,r+2}(t_f) & \cdots & m_{1,n}(t_f) \\
    m_{2,r+1}(t_f) & m_{2,r+2}(t_f) & \cdots & m_{2,n}(t_f) \\
    \vdots & \vdots & \ddots & \vdots \\
    m_{n-r,r+1}(t_f) & m_{n-r,r+2}(t_f) & \cdots & m_{n-r,n}(t_f)
\end{bmatrix}
\begin{bmatrix}
    y_{r+1}(t_0) \\
    y_{r+2}(t_0) \\
    \vdots \\
    y_n(t_0)
\end{bmatrix}
= \begin{bmatrix}
    y_{r+1}(t_0) \\
    y_{r+2}(t_0) \\
    \vdots \\
    y_n(t_0)
\end{bmatrix}
\]