On the Rate of Convergence of Some Feasible Direction Algorithms

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Abstract. This paper is concerned with first-order methods of feasible directions. Pironneau and Polak have recently proved theorems which show that three of these methods have a linear rate of convergence for certain convex problems in which the objective functions have positive definite Hessians near the solutions. In the present note, it is shown that these theorems on rate of convergence can be extended to larger classes of problems. These larger classes are determined in part by certain second-order sufficiency conditions, and they include many nonconvex problems. The arguments used here are based on the finite-dimensional version of Hestenes' indirect sufficiency method.

Key Words. Mathematical programming, nonlinear programming, inequality constraints, numerical methods, descent methods.

1. Introduction

Let \( f_0, f_1, \ldots, f_m \) be real-valued, continuously differentiable functions on real Euclidean \( n \)-space \( R^n \). Let

\[
S = \bigcap_{k=1}^{m} \{ x \in R^n : f_k(x) \leq 0 \},
\]

and let \( P \) refer to the problem of minimizing \( f_0 \) over \( S \). One popular way to obtain a numerical solution to \( P \) is to use a method of feasible directions. A number of methods of feasible directions were developed by Zoutendijk and are described in Ref. 1. A recent, detailed discussion of methods of feasible directions appears in Polak (Ref. 2).
In this paper, we shall examine three methods of feasible directions. We are concerned with questions about rate of convergence, rather than with questions about convergence alone. Pironneau and Polak have shown (Ref. 3) that these three methods do often converge at a linear rate. Their results apply to certain convex programs. The purpose of the present note is to show how the convexity assumptions can be weakened considerably. Our assumptions are defined mainly by certain second-order sufficiency conditions for optimality (see Ref. 4, p. 37, or Ref. 5, pp. 26–30). In obtaining our results, we have begun with the arguments of Pironneau and Polak (Ref. 3). We have then modified these arguments so as to obtain arguments which remain valid under our weaker hypotheses. The modifications are carried out by use of techniques like those employed elsewhere by the author (Ref. 6); these techniques are based on the finite-dimensional version of Hestenes' indirect sufficiency method (Ref. 4, p. 37 and pp. 151–167). Also, we obtain estimates for the important constants which are slightly better than those given by Pironneau and Polak.

We present the algorithms and some preliminaries in Section 2. We discuss one method of Zoutendijk in Section 3 and two methods of Pironneau and Polak in Section 4. The paper concludes with some miscellaneous remarks.

2. Algorithms and Preliminaries

For ease of reference, we list certain assumptions under the following hypothesis.

**Hypothesis 2.1.** Assume that \( f_0, f_1, \ldots, f_m, S, \) and Problem P are as above. Assume that \( x_0 \) is a point in \( S \) such that the set

\[
\{x \in S : f_0(x) \leq f_0(x_0)\}
\]

is compact. Assume that \( B \) is a compact, convex neighborhood of the origin \( O \) in \( \mathbb{R}^n \).

**Remark 2.1.** From now on, we shall always assume that Hypothesis 2.1 is fulfilled. It follows that \( f_0 \) attains a minimum on \( S \).

**Remark 2.2.** We denote by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \), respectively, the Euclidean inner product and norm in finite-dimensional Euclidean spaces. If \( E \subset \mathbb{R}^n \), we let \( \text{co}(E) \) be the convex hull of \( E \). If \( F \) is a real-valued function on \( \mathbb{R}^n \),