The Drazin Inverse and a Spectral Inequality of Marcus, Minc, and Moyls\textsuperscript{1}

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Abstract. Some properties of the Drazin inverse of a square matrix are derived which, together with elementary divisor theory, are used to investigate a spectral inequality which relates the maximal eigenvalues of two nonnegative indecomposable matrices.

Key Words. Drazin inverse, primitive matrix, nonnegative matrix, computing methods, linear systems.

1. Introduction

All matrices are assumed to be square. A nonnegative indecomposable matrix $A$ is called primitive if $A$ contains only one eigenvalue $\lambda$ (algebraic multiplicity one) whose modulus is equal to the spectral radius of $A$. An equivalent condition to being primitive is for $A$ to be nonnegative and $A^n$ to be positive for some natural number $n$ (Ref. 1, p. 128).

The main purpose of this paper is to exhibit some properties possessed by the Drazin inverse $A^D$ (Ref. 2) of certain matrices $A$ and to apply these properties to refine a spectral inequality of Marcus, Minc, and Moyls (Ref. 3) which relates the maximal eigenvalues of two nonnegative indecomposable matrices. We show that $A^D$ is a polynomial function of $A$ and apply the theory of elementary divisors to compute $A^D$. This approach is somewhat similar to one used recently by Lovass-\textsuperscript{2}

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Nagy and Powers (Ref. 4) to establish a relation between the Moore-Penrose and commuting reciprocal inverses. However, their approach considers $A^D$ only for the case when $A$ has index (Ref. 5) zero or one (that is, $A^D$ is the CR inverse for $A$ in this case). $A^T$ will denote the transpose of $A$.

2. Properties of the Drazin Inverse

**Theorem 2.1.** If $A$ is a primitive matrix, then $A^D$ is primitive iff $A^D$ is nonnegative.

**Proof.** Only the sufficiency needs establishing. Suppose that $A$ is of index $t$ and $A^p$ is positive. If $k = \max\{t, p\}$, then $A^D A^{k+1}$ is positive. Since $A^D$ contains no zero rows or columns, $(A^D)^p$ is positive, which shows that $A^D$ is primitive.

**Corollary 2.1.** If $A$ is primitive of index one and $A^D$ is nonnegative, then $A^D$ is a primitive spectral inverse (Ref. 6) of $A$.

The following theorem is not new and a proof is given in Ref. 7, p. 80. However, the proof given below is constructive, as opposed to the one in Ref. 7, and indicates a new method for computing the Drazin inverse.

**Theorem 2.2.** The Drazin inverse of $A$ is a polynomial in $A$ over the complex numbers.

**Proof.** Since we can write

$$A = P \begin{bmatrix} B & 0 \\ 0 & N \end{bmatrix} P^{-1},$$

where $B$ is nonsingular and $N$ is nilpotent of index $k$, then

$$A^D = P \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Therefore, showing that $A^D$ is a polynomial in $A$ is equivalent to showing that

$$X = \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

is a polynomial in

$$Y = \begin{bmatrix} B & 0 \\ 0 & N \end{bmatrix}.$$