Second-Order Necessary Conditions in Optimal Control: Accessory-Problem Results without Normality Conditions

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Abstract. An optimal control problem, which includes restrictions on the controls and equality/inequality constraints on the terminal states, is formulated. Second-order necessary conditions of the accessory-problem type are obtained in the absence of normality conditions. It is shown that the necessary conditions generalize and simplify prior results due to Hestenes (Ref. 5) and Warga (Refs. 6 and 7).

Key Words. Optimal control, second-order necessary conditions, accessory problem, control constraints, normality condition.

1. Introduction

Second-order theory for problems in optimal control has been studied for some time. The earliest results were motivated by the work of Bliss (Ref. 1) in the calculus of variations. They concerned problems with open control sets and depended on heuristic arguments to obtain both necessary conditions and local sufficiency conditions; see, for example, Refs. 2–4, which include some additional references. Mathematically rigorous treatments of second-order necessary conditions seem to be limited (Refs. 5–8).

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Hestenes, whose work is the earliest, considered a fairly general optimal control problem, but made the standard assumption that the control set is open. His main result (Theorem 9.1, Chapter 6, Ref. 5) states that the second variation of a suitably defined function is nonnegative on a set of admissible variations related to the first-order necessary conditions. More recently, Warga (Ref. 6) obtained a similar result, stated in a somewhat different way, for problems where the controls are restricted to a convex, not necessarily open, constraint set. A more radical departure is considered in Refs. 7 and 8, where relaxed controls are introduced. The resulting second-order conditions may be interpreted as exploiting strong (Weierstrass–Pontryagin) variations instead of the weak (Lagrange) variations used in Refs. 5 and 6. It should be emphasized that all of the above conditions are of the accessory-problem type and are global over time. Thus, they are quite different from the pointwise-in-time, second-order conditions considered in the theory of singular control (Refs. 9–12). See Ref. 7 for some additional remarks on these matters.

The approaches to the proofs in Refs. 5–8 appear philosophically different. Hestenes works directly in the space of state-control pairs, while Warga obtains necessary conditions for an abstract optimization problem, which he then applies to the optimal control problem. There is, however, a common ground. Both modes of proof rely on the implicit function theorem to generate a one-parameter family of admissible elements. This is done in such a way that the range of a variational operator, which is associated with problem constraints, must have full dimension. Conditions which imply the full-range property are called normality conditions, because they imply (among other things) that the multiplier associated with the cost in the necessary conditions is nonzero.

Normality conditions, such as those found in Refs. 5–8, are unpleasant because they are usually difficult to verify and may, on occasion, fail to be satisfied. This is why first-order conditions which do not require normality, such as those due to Fritz John (in mathematical programming) and Pontryagin (in optimal control), are popular. In the absence of normality conditions, there are some second-order necessary conditions in the mathematical programming literature (Refs. 13–16); but, apart from the singular control results (Refs. 9, 11, 12), there seem to be none in the theory of optimal control. Our objective in this paper is to make significant progress in filling this gap.

We begin in Section 2 by stating a rather general optimal control problem which requires weak (not necessarily convex) assumptions on the control set and includes mixed equality and inequality constraints on the initial and terminal states. This problem subsumes the problems in Refs. 5–8 as special cases. Our initial attention (Section 3) is on the case of weak