Liapunov Reachability and Optimization in Control¹

R. J. Stoner²

Communicated by G. Leitmann

Abstract. In this paper, we find sufficient conditions that do not rely upon any Liapunov stability results for the reachability of the target set by an admissible control from an a priori specified set of states. The conditions found are less restrictive than those obtained by Sticht, Vincent, and Schultz in Ref. 1.

For control problems with specific integral performance index, we also show that, if a quantitative Liapunov function for optimization exists, then it may also serve as a qualitative Liapunov function ensuring the reachability of the target. The results are illustrated by examples.

Key Words. Liapunov function, reachability, optimization, quantitative control, qualitative control.

1. Introduction

The underlying assumption that occurs in much of the work done in optimal control is that there exists an admissible control which drives the state to a desired target set (controllability of the state). This reachability of the target set from a given initial state, as a problem itself, is of considerable importance. Also in the last two decades, there has been a significant amount of research on the application of Liapunov's qualitative second method in a variety of problems, particularly in nonlinear mechanics and control systems. A great amount of this work has been concerned with the various forms of Liapunov stability and its application to the design of nonlinear systems; see, for example, Refs. 2 and 3. However, over the last few years, there has been a resurgence of the qualitative method in the

¹ The author is indebted to Prof. J. M. Skowronsaki and Prof. P. Seibert for their useful comments and discussions.
² Senior Tutor, Mathematics Department, University of Queensland, Brisbane, Queensland, Australia.
study of controllability to a target set, controllability with capture in the target, and even to the study of qualitative differential games (cf. Refs. 1 and 4–6).

In this paper, we propose sufficient conditions for the reachability of the target set by admissible controls via the Liapunov method which are less stringent in terms of continuity requirements than in known results (cf. Ref. 1).

In conclusion, we investigate the relationship between reachability and optimization. It is known that Bellman’s equation, which is a necessary condition for optimality, has a solution which, if it exists, under certain conditions serves as a Liapunov function for the stability of the controlled system; see, for example, La Salle’s notes (Ref. 7, ordinary differential equations) and Kopp (Ref. 8, partial differential equations). In a similar fashion to the work that has been done with Liapunov stability and optimization, we establish a joint sufficiency theorem for both reachability and optimization in Section 6.

2. Notation

\[ R^+ = \text{positive real axis, time axis}; \]
\[ R^* = \text{extended positive real axis}; \]
\[ R^n = \text{real } n\text{-dimensional Euclidean space with norm } \| \cdot \|_n; \]
\[ \chi(R^n) = \text{space of all nonempty compact subsets of } R^n \text{ with usual Hausdorff topology}. \]

For \( X, \theta \subseteq R^n \), we define:

\[ C(X) = \text{complement of } X \text{ in } R^n; \]
\[ \partial \theta = \text{boundary set of } \theta; \]
\[ \overset{\circ} X = \text{interior of } X; \]
\[ \bar{X} = \text{closure of } X; \]
\[ X \setminus \partial = \{ x \mid x \in X \text{ and } x \notin \partial \}; \]
\[ \mathcal{C}^1(X; R^+) = \text{space of all functions mapping } X \text{ to } R^+ \text{ with continuous first derivatives}. \]

3. Control Problem

**Formulation.** We assume that the controlled object is a system governed by the autonomous differential equation

\[ \dot{x} = f(x, \sigma), \quad (1) \]