Remarks on a Simple Optimal Control Problem with Monotone Control Functions

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Abstract. We consider the minimization of the mean-square deviation of a prescribed function from the class of monotone functions. Two problems are considered. The first problem places no restriction on the initial value of the controls, while the second problem assumes that all the control functions must start at a fixed initial value. Optimal controls are exhibited in both problems. Finally, we consider the situation with general payoff and dynamics and give the heuristic characterization of the value function for such problems.

Key Words. Monotone control functions, fixed initial control, projection on convex set, monotone Lipschitz functions, nonlinear control problems.

1. Introduction

Reid in Ref. 1 considered the problem of optimal control (really, calculus of variations) in the case that the controls are required to be monotone. As an example, he considered the problem with payoff functional

\[ P(y) = \int_t^T (y(s) - d(s))^2 \, ds, \]

where \( d(s) \) on \([t, T]\) is a given bounded, measurable function and \( y(s) \) is required to be monotone nonincreasing on \([t, T]\). By an involved proof, he showed that the optimal control \( c(s) \) is the derivative a.e. of the least concave majorant of the function

\[ D(s) = \int_t^s d(r) \, dr. \]
In this paper, we present a new and more direct proof of this result. Furthermore, we restrict the class of control functions to those which start at some initial point which is fixed, and we derive the optimal control. Clearly, it will not, in general, be the derivative of the least concave majorant of \( D \), since the initial control position may not have the correct magnitude.

This problem has some application to economics, if we interpret the controls \(-y(s)\) as the cumulative investment to meet the demand \( d(s)\) with no option of disinvestment. Of course, the payoff functional here is not quite realistic. The stochastic counterpart for a more restrictive payoff was considered by Benes, Shepp, and Witsenhausen (Ref. 2), and it is interesting to compare the results. Restricting the controls to start at a fixed position is perhaps more realistic for applications.

The last section reviews the results of Barron and Jensen (Ref. 3), providing a heuristic reasoning for the characterization of the value function. The results indicate that the value function does not satisfy an equation, but a system of inequalities analogous to the Kuhn–Tucker inequalities. The remaining results of Reid are contained in this section and rederived in the context of Ref. 3.

In summary, the main results of this paper consist of a new and simpler proof of the main result of Ref. 1, along with an extension to the fixed initial control position case. It is to be noted that optimal monotone controls are not easy to come by, and it is hoped that the results herein may shed some light on some of the difficulties involved.

2. Quadratic Payoff Case

Let \( K \) be the class of monotone nonincreasing functions on \([t, T]\) in the Hilbert space \( L^2[t, T] \) of square-integrable functions. Clearly, \( K \) is a convex cone in \( L^2 \). \( K \) is also closed in \( L^2 \) since, if \( v_n \in K \) with \( v_n \rightarrow v \) in \( L^2 \), then \( v_n \) has a subsequence converging pointwise almost everywhere to \( v \). Thus, \( v \) is nonincreasing. It is well known (Balakrishnan, Ref. 4) that, for each \( x \) in \( L^2 \), there is a unique \( k_0 \) in \( K \) so that

\[
\|k_0 - x\| = \min_{K} \|k - x\|, 
\]

and \( k_0 \) is the projection of \( x \) on \( K \). A necessary and sufficient condition for \( k_0 \) in \( K \) to be the projection on \( K \) of \( x \) is that

\[
(k_0, y - k_0) \geq (x, y - k_0), 
\]

for every \( y \) in \( K \). Here, \((\cdot, \cdot)\) is the inner product in \( L^2 \). That condition (1)