Optimum Design of One-Dimensional Journal Bearings

G. T. McAllister and S. M. Rohde

Communicated by G. Leitmann

Abstract. When two surfaces in relative motion are separated by a thin layer of viscous fluid, they are kept apart by the pressure generated in the fluid film due to viscous forces. The magnitude of the resultant of this pressure is the load capacity of the bearing. We find the shape of a one-dimensional journal bearing which supports the greatest load.

Key Words. Distributed control problems, hydrodynamic lubrication theory, journal bearings, optimum design.

1. Introduction

Hydrodynamic lubrication theory is concerned with the separation of two surfaces in relative motion by a thin layer of viscous fluid. The surfaces are kept apart by pressure generated in the fluid film due to viscous forces. The shape, extent, and thickness of the fluid film determines, to a large extent, the magnitude of the resultant of this pressure, the load capacity of the bearing. Our objective is to find the shape of a steadily loaded journal bearing, operating with an incompressible isoviscous lubricant, which has the greatest load capacity for a given minimum film thickness.

The family of admissible shapes (or profiles) for our bearings will be denoted by $\mathcal{H}$, where

$$\mathcal{H} = \{H(x): h \in L_\infty((0, 2\pi)), 1 \leq H(x) \leq M\},$$

(1)

$M > 1$ is given, and the minimum film thickness is taken to be one. For each $H \in \mathcal{H}$, the pressure distribution $p_H(x)$ under the bearing shape $H$ is

1 This work was partially supported by the National Science Foundation.
2 Professor, Center for the Application of Mathematics, Lehigh University, Bethlehem, Pennsylvania.
3 Senior Staff Research Scientist, Power Systems Research Department, General Motors Research Laboratories, Warren, Michigan.

599
given by the equation
\[ \int_0^{2\pi} \{H^3(x)p'_H(x) - H(x)\}z'(x) \, dx = 0, \] (2)
which is to hold for all
\[ \zeta \in H^1_{2,0}((0, 2\pi)) \quad \text{and} \quad p_H \in H^1_{2,0}((0, 2\pi)); \]
see Morrey (Ref. 1) for a complete description of this space. The load-bearing capacity of a journal bearing with bearing shape \( H \) is given by the functional
\[ \mathcal{L}[H] = \left( \int_0^{2\pi} p'_H(x) \sin x \, dx \right)^2 + \left( \int_0^{2\pi} p'_H(x) \cos x \, dx \right)^2 \] (3)
We solve the following problem: Find a bearing shape \( H_0 \in \mathcal{H} \) such that
\[ \mathcal{L}[H_0] = \max \{\mathcal{L}[H] : H \in \mathcal{H}\}. \] (4)
Clearly, we solve this problem if we replace \( \mathcal{L}[H] \) in (4) by \( \mathcal{W}[H] \) with
\[ \mathcal{W}[H] = (\mathcal{L}[H])^2. \]
This we will do in the sequel as it simplifies many computations.

When \( \mathcal{L}[H] \) is taken as the integral of \( p_H(x) \) over \([0, 2\pi]\), we have in (4) the problem of finding the slider bearing profile which supports the largest load. That problem was solved in closed form in Ref. 2, but those methods do not apply to the complicated functional in (3).

In Section 2, we prove that (4) has a solution; and, since the first variation of (3) evaluated at \( H_0 \) is zero, we show that the range of an optimizing profile \( H_0 \) must be contained in a set of three values, but we have no knowledge of the nature of those three measurable subsets of \([0, 2\pi]\) over which \( H_0 \) assumes one of these values. This suggests an auxiliary problem of finding a Lebesgue measurable subset \( E_0 \subset [0, 2\pi] \) such that
\[ W(E_0) = \sup \{W(E) : E \in \mathcal{M}_\lambda\}, \]
where \( \mathcal{M}_\lambda \) is the set of all measurable subsets \( E \) of \([0, 2\pi]\) whose Lebesgue measure \( m(E) \) is the given value \( \lambda \in (0, 2\pi) \) and
\[ W(E) = \left( \int_E \sin x \, dx \right)^2 + \left( \int_E \cos x \, dx \right)^2. \]
In Section 3, we find that \( E_0 \) is an interval and that
\[ W(E_0) = 4 \sin^2(\lambda/2). \]
These results are extended to cover the general case in Section 4, where we reduce the problem in (4) to a simple problem of the calculus.