Reflections on Nondifferentiable Optimization, 
Part 1, Ball Gradient¹

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Abstract. In considering the nondifferentiable optimization problem, a new concept is introduced, known as the ball gradient. The ball-gradient magnitude is positive at any local minimum point, independently of whether the minimum point is well behaved, a cusp, or a sheet minimum. The ball-gradient magnitude is negative outside a ball of radius € around a local minimum point and thus is usable as a terminating criterion on nondifferentiable functions.

Key Words. Nondifferentiable optimization, subgradients, ball gradients.

1. Introduction

In Ref. 1, the concept of a well-behaved function for unconstrained minimization is introduced. Such a function was defined as satisfying four conditions: (i) a lower bound L exists; i.e., $F(x) \geq L$, for all $x \in S$; (ii) the Hessian matrix $G$ exists at all points $x \in S$ and is uniformly bounded; i.e., for all $||z|| = 1$,

$$-M \leq z^T G(x) z \leq M;$$

(iii) the third derivatives of $F(x)$ exist at all $x \in S$; and (iv) a local minimum point $x^*$ exists, and the Hessian at $x^*$ is well conditioned; i.e., for all $||z|| = 1$,

$$0 < m \leq z^T G(x^*) z \leq M.$$ 

The problem of nondifferentiable optimization occurs when conditions (ii), (iii), (iv) are no longer satisfied. In particular, the gradient vector $g$ and,

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hence, the Hessian matrix are not defined at all points $x \in S$. It will be convenient to distinguish three classes of nondifferentiable optimization problems. The author first heard these classes defined in the presentation given by Wolfe at the conference on optimization held at the University of Dundee in 1972 (Ref. 2). In all three classes of problems, it will be assumed that: (i) a local minimum point $x^*$ exists; and (ii) the contours $F(x) = F(x^*) + \delta$ are closed for sufficiently small $\delta$.

Class A. *Well-Behaved Minimum Point.* A minimum is said to be well behaved if there exists $L_1 > F(x^*)$ and if the function is well behaved in the set $S$, where

$$S = \{x : F(x) < L_1\}.$$

**Remark 1.1.** This implies that all nondifferentiable points occur away from the neighborhood of the minimum.

Class B. *Cusp Minimum.* A local minimum point $x^*$ is said to be a cusp if the gradient vector $g$ is not defined at $x^*$ and this is an isolated nondifferentiable point. We may typify such a function in the region of $x^*$ by

$$F(x) = F(x^*) + (\frac{1}{2}x^TAx + O\|x\|^2)^v, \quad v < 1,$$

and note that the contours of this function are again approximately ellipsoidal in the region of $x^*$. In Eq. (1), the origin of the coordinate system is assumed to be at $x^*$.

Class C. *Sheet Minimum.* A local minimum point is said to be a sheet minimum, if the gradient vector $g$ is not defined at $x^*$ nor is it defined on one or more continuous sheets passing through $x^*$. Typically, $F$ may be defined in terms of a set $\Phi^k, k \in K$, as

$$F(x) = \max_{k \in K} \Phi^k(x);$$

then, the nondifferentiable points occur at the intersections

$$F(x) = \Phi^k(x) = \Phi^J(x), \quad k \neq J.$$

The typical sheet minimum will occur at the intersection of a number of such sheets.

**Remark 1.2.** The properties of these three classes of minima are very different, and the modifications, that they require in an optimization algorithm designed for well-behaved functions may well be different.