A Minimization Method for the Sum of a Convex Function and a Continuously Differentiable Function\textsuperscript{1}

H. Mine\textsuperscript{2} and M. Fukushima\textsuperscript{3}

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Abstract. This paper presents a method for finding the minimum for a class of nonconvex and nondifferentiable functions consisting of the sum of a convex function and a continuously differentiable function. The algorithm is a descent method which generates successive search directions by solving successive convex subproblems. The algorithm is shown to converge to a critical point.

Key Words. Mathematical programming, nonconvex nondifferentiable optimization problems, subgradients, critical points.

1. Introduction

This paper presents a method for finding the minimum for a class of nonconvex and nondifferentiable functions. The problem is as follows:

$$\text{minimize } \phi(x) = f(x) + g(x), \quad \text{over } x \in \mathbb{R}^n,$$

where

$$g: \mathbb{R}^n \rightarrow (-\infty, +\infty]$$

is a closed, proper, convex, but not necessarily differentiable, function and

$$f: \mathbb{R}^n \rightarrow (-\infty, +\infty]$$

is a function which is continuously differentiable on an open set containing \(\text{dom } g\), but \(f\) need not be convex. Problem (1) contains a fairly wide class of

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\textsuperscript{2}Professor, Department of Applied Mathematics and Physics, Faculty of Engineering, Kyoto University, Kyoto, Japan.

\textsuperscript{3}Graduate Student, Department of Applied Mathematics and Physics, Faculty of Engineering, Kyoto University, Kyoto, Japan.
problems which are encountered in practice. It may be noted that the function \( \phi \) is semismooth, good, semiconvex, quasidifferentiable in the sense of Mifflin (Ref. 1), Feuer (Ref. 2), Vainberg (Ref. 3), Pshenichnyi (Ref. 4), respectively, and Clarke differentiable (Ref. 5), if it is finite everywhere.

If the function \( \phi \) happens to be convex, as is true if \( f \) is convex, various algorithms for nondifferentiable convex optimization problems (see, e.g., Refs. 6-10) may be applied. Most of those methods utilize the subgradients or \( \varepsilon \)-subgradients of convex functions, both of which are described in Rockafellar (Ref. 11). Recently, Feuer (Refs. 2 and 12), Goldstein (Ref. 13), and Mifflin (Ref. 14) have used the Clarke generalized gradient (Ref. 5) to generalize these techniques to apply to various classes of nonconvex, nondifferentiable objectives.

The algorithm under consideration here is a descent method which generates successive search directions by solving successive convex subproblems, each of which may be regarded as an approximation of the original problem. The present algorithm is particularly useful if \( g \) is simple enough to be minimized without the aid of general-purpose convex minimization algorithms such as those in Refs. 7-10. For example, see Examples 1.1 and 3.1 below.

Although we are mainly concerned with the unconstrained problem (1), the present method can be applied to constrained problems, as shown in the following examples.

Example 1.1. Consider the following problem:

\[
\text{minimize } f(x), \\
\text{subject to } x \in S \subset \mathbb{R}^n,
\]

where \( f: \mathbb{R}^n \to (-\infty, +\infty) \) is a continuous differentiable function and \( S \) is a closed, convex set. Then, problem (2) may be rewritten in the equivalent form:

\[
\text{minimize } f(x) + \delta_S(x), \quad \text{over } x \in \mathbb{R}^n,
\]

where \( \delta_S \) is the indicator function of \( S \) (Ref. 11) defined by

\[
\delta_S(x) = 0, \quad \text{if } x \in S, \\
\delta_S(x) = +\infty, \quad \text{otherwise}.
\]

Clearly, problem 3 is a problem of the form (1).