Convergence Properties of Local Solutions of Sequences of Mathematical Programming Problems in General Spaces \(^1,2\)

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Abstract. This paper gives several sets of sufficient conditions that a \textit{local} solution \(x^k\) exists of the problem \(\min_{R^k} f^k(x), k = 1, 2, \ldots\), such that \(\{x^k\}\) has cluster points that are \textit{local} solutions of a problem of the form \(\min_R f(x)\). The results are based on a well-known concept of topological, or \textit{point-wise} convergence of the set \(\{R^k\}\) to \(R\). Such results have been used to construct and validate large classes of mathematical programming methods based on successive approximations of the problem functions. They are also directly applicable to parametric and sensitivity analysis and provide additional characterizations of optimality for large classes of nonlinear programming problems.

\textbf{Key Words.} Nonlinear programming, approximation theory, nonconvex programming, perturbation methods, sensitivity analysis.

\(^1\) Results closely related to those obtained here were presented in \textit{Some Properties of Convergent Sequences of Mathematical Programs} by P. D. Robers and the author to the 36th National Meeting of the Operations Research Society of America, Miami, Florida, 1969. The author deeply acknowledges the contributions made by Dr. Robers in developing these earlier results.

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1. Introduction

The motivation for the present paper is an inquiry into the approach of solving a mathematical programming problem
\[
\text{minimize } f(x) \text{ subject to } x \in R \quad (P)
\]
by solving a sequence of approximations
\[
\text{minimize } f^k(x) \text{ subject to } x \in R^k \quad (P^k)
\]
of Problem P, for \( k = 1, 2, 3, \ldots \).

This approach is a classical one and has appeal if the auxiliary problems are considerably easier to solve than the original problem, and if a sequence of solutions of the approximation problems converges satisfactorily to a solution of the given problem.

If attention is focused on the relationship of the problems \( \{P^k\} \) to Problem P, rather than on any particular method of approximation, questions arise as to precisely when and in what sense a sequence of local solutions of the approximation problems converges to a local solution of the given problem. The analysis in this paper is based on the requirement that the approximation problems \( \{P^k\} \) themselves converge, in some suitable sense, to Problem P. Various conditions are then given that lead to limiting characterizations of a sequence \( \{x^k\} \) of local minimizing points of the problems \( \{P^k\} \).

Such results can be used to devise and validate procedures for solving Problem P by defining an appropriately convergent approximation sequence of problems \( \{P^k\} \) that are solvable by known methods. Examples of procedures explicitly based on solving a sequence of approximation problems are numerous, e.g., the Raleigh–Ritz method of the variational calculus (Refs. 1–2), penalty methods (Ref. 3), methods of centers (Ref. 4), and cutting-plane methods (Ref. 5). In fact, most algorithms for mathematical programming can be characterized and validated from this point of view [e.g., see Zangwill (Ref. 6) and Meyer (Ref. 7)].

Other potential applications are the obtaining of neighborhood and asymptotic characterizations of optimality for Problem P and the derivation of conditions associated with a perturbation (sensitivity) or parametric analysis of the given problem, e.g., when Problem P has the form: minimize \( f[x, \alpha(k)] \) subject to \( x \in R[\alpha(k)] \). Examples of such results appear in Refs. 8–10.

Our conditions are intimately related to some well-known results [see Berge (Ref. 11), for example] and more recent results obtained by Dantzig, Folkman, and Shapiro (Ref. 8), Meyer (Ref. 7), Mosco (Ref. 12), Evans and Gould (Ref. 9), and Greenberg and Pierskalla (Ref. 10).