Variational Perturbations of the Linear–Quadratic Problem

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Abstract. A sequence of perturbations
\[ \dot{x} = A_n x + B_n u, \quad \|u\|_{L^2} \leq 1, \quad (P_n) \]
\[ x(0) = x^0, \quad n = 0, 1, 2, 3, \ldots \]
is given of the linear–quadratic optimal control problem consisting of minimizing
\[ \int_0^1 ((u - \bar{u})^T(u - \bar{u}) + (x - \bar{x})^T(x - \bar{x})) \, dt, \]
subject to \((P_0)\). We assume that \(\{A_n\}\) bounded in \(L^1\) and \(\{B_n\}\) is bounded in \(L^2\). Then, a necessary and sufficient condition so that, for every \(\bar{u}\), \(\bar{x} \in L^2\), and for every \(x^0\), the optimal control for \((P_n)\) converges strongly in \(L^2\) to the optimal control for \((P_0)\) and the optimal state converges uniformly is that \(A_n \to A_0\) weakly in \(L^1\) and \(B_n \to B_0\) strongly in \(L^2\).

Key Words. Linear–quadratic problem, stability analysis, variational perturbations.

1. Introduction

Consider a linear control process described by the differential system
\[ \dot{x}(t) = A_0(t)x(t) + B_0(t)u(t) \quad \text{a.e. in } [0, 1], \]
\[ x(0) = x^0, \tag{1} \]
and a sequence of perturbations described by
\[ \dot{x}(t) = A_n(t)x(t) + B_n(t)u(t) \quad \text{a.e. in } [0, 1], \]
\[ x(0) = x^0. \tag{2} \]

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Here, \( x^0 \in \mathbb{R}^p \); \( u \in L^2[0, 1] \) is a vector-valued function of dimension \( q \); and \( \|u\|_{L^2} \leq 1 \). For \( n \geq 0 \), \( A_n \in L^1[0, 1] \) is a square matrix of dimension \( p \) and \( B_n \in L^2[0, 1] \) is of dimension \( p \times q \).

We shall denote by \( x_n(u) \) the state corresponding to the control \( u \). Then,

\[
x_n(u)(t) = \frac{d}{dt} x(t) + \int_0^t F_n^{-1} B_n u \, ds,
\]

where

\[
F_n(t) = A_n(t) F_n(t) \quad \text{a.e. in } [0, 1],
\]

\[
F_n(0) = I_p = \text{identity matrix of dimension } p,
\]

that is,

\[
F_n(t) = I_p + \int_0^t A_n F_n \, ds, \quad 0 \leq t \leq 1.
\]

For given \( x^0 \in \mathbb{R}^n \) and \( \bar{u}, \bar{x} \in L^2[0, 1] \), we set

\[
J_n(u) = \int_0^1 [(u - \bar{u})^T (u - \bar{u}) + (x_n(u) - \bar{x})^T (x_n(u) - \bar{x})] \, dt, \quad n \geq 0. \tag{3}
\]

We consider, for all \( n \), the linear–quadratic problem consisting of the minimization of (3), subject to the constraint

\[
\|u\|_{L^2} \leq 1.
\]

It is well known that, for \( n = 0, 1, 2, \ldots \), and for any \( x^0, \bar{u}, \bar{x} \), there is a unique optimal control \( \bar{u}_n \).

In this paper, we shall prove the following result.

**Theorem 1.1.** Assume that

\[
\sup_{n \geq 0} \|A_n\|_{L^1} \leq H < +\infty, \quad \sup_{n \geq 0} \|B_n\|_{L^2} \leq K < +\infty. \tag{4}
\]

Then, the following facts (a), (b) are equivalent:

(a) for all \( \bar{u}, \bar{x} \in L^2[0, 1] \) and for all \( x^0 \in \mathbb{R}^p \), (i) \( J_n(\bar{u}_n) \) converges to \( J_0(\bar{u}_0) \), and (ii) \( \bar{u}_n \) converges strongly in \( L^2[0, 1] \) to \( \bar{u}_0 \), \( x_n(\bar{u}_n) \) converges uniformly in \( [0, 1] \) to \( x_0(\bar{u}_0) \);

(b) \( A_n \) converges weakly in \( L^1[0, 1] \) to \( A_0 \) and \( B_n \) converges strongly in \( L^2[0, 1] \) to \( B_0 \).

Stated in different words, if the hypotheses (4) are satisfied, a necessary and sufficient condition that the optimal controls and the corresponding