Sufficient Optimality Conditions and Duality for a Quasiconvex Programming Problem

C. R. Bector, S. Chandra, and M. K. Bector

Communicated by G. Leitmann

Abstract. Under differentiability assumptions, Fritz John Sufficient optimality conditions are proved for a nonlinear programming problem in which the objective function is assumed to be quasiconvex and the constraint functions are assumed to quasiconcave/strictly pseudoconcave. Duality theorems are proved for Mond-Weir type duality under the above generalized convexity assumptions.

Key Words. Fritz John sufficient optimality conditions, Mond-Weir duality.

1. Introduction

On the lines of Mond and Weir (Ref. 1), consider the following nonlinear programming problems:

(P) \( \text{minimize } f(x), \quad x \in S \) \hfill (1)

(PE) \( \text{minimize } f(x), \quad x \in X \) \hfill (2)

where

(i) \( S = \{ x; x \in X^0, g(x) \geq 0 \} \)

\[ = \{ x; x \in X^0, g_i(x) \geq 0, i \in M \}, \] \hfill (3)

1 The first author is thankful to the Natural Science and Engineering Research Council of Canada for financial support through Grant No. A-5319. The authors are thankful to Professor B. Mond for suggestions that improved the original draft of the paper.
2 Professor, Department of Actuarial and Management Sciences, University of Manitoba, Winnipeg, Manitoba, Canada.
3 Assistant Professor, Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi, India.
4 Graduate Student, Department of Actuarial and Management Sciences. University of Manitoba, Winnipeg, Manitoba, Canada.

209

0022-3239/88/1100-0209$06.00/0 © 1988 Plenum Publishing Corporation
(ii) \( X = \{ x; x \in X^0, g(x) \geq 0, h(x) = 0 \} \)
\( = \{ x; x \in X^0, g_i(x) \geq 0, i \in M, h_k(x) = 0, k \in K \}, \)  
\( (4) \)

(iii) \( M = \{ 1, 2, \ldots, m \}, K = \{ 1, 2, \ldots, k \}, \) 
\( (5) \)

(iv) \( X^0 \) is an open set of \( \mathbb{R}^n, \)
\( (v) \) \( f : X^0 \to \mathbb{R}, g : X^0 \to \mathbb{R}^m \) and \( h : X^0 \to \mathbb{R}^k \) are differentiable functions.

Evidently, if \( K = \emptyset \) (null set), \( (PE) \) becomes \( (P) \).

Sufficient optimality conditions for such problems are important theoretically as well as computationally and are studied by Mangasarian (Ref. 2), Bector and Grover (Ref. 4), Bector and Gulati (Ref. 5), Singh (Ref. 6), and Skarpness and Sposito (Ref. 7). Mangasarian (Ref. 2), assuming \( f \) to be pseudoconvex, \( g_i \) [where \( I = \{ i; g_i(\bar{x}) = 0 \} \)] to be quasiconcave and \( h \) to be both quasiconvex and quasiconcave at \( \bar{x} \in X \), showed that, if \( (\bar{x}, \bar{y}, \bar{z}) \) is a solution to the following Kuhn-Tucker-type conditions:

\[
\nabla [f(x) - y^t g(x) - z^t h(x)] = 0, \quad (6)
\]
\[
y^t g(x) = 0, \quad (7)
\]
\[
g(x) \geq 0, \quad (8)
\]
\[
h(x) = 0, \quad (9)
\]
\[
y \geq 0, \quad y \in \mathbb{R}^m, \quad z \in \mathbb{R}^k, \quad (10)
\]

then \( \bar{x} \) is \( (PE) \)-optimal.

Bhatt and Misra (Ref. 3), assuming all of \( f, g, h \) to be convex at \( \bar{x} \in X \), showed that the above conditions \( (6)-(10) \), with the additional restriction \( z \geq 0 \), are sufficient for \( \bar{x} \) to be \( (PE) \)-optimal.

Assuming \( f \) to be convex and \( g \) to be strictly concave at \( \bar{x} \in S \), Mangasarian (Ref. 2) showed that, if \( (\bar{x}, \bar{y}_0, \bar{y}) \) is a solution to the following Fritz John-type conditions:

\[
\nabla [y_0 f(x) - y^t g(x)] = 0, \quad (11)
\]
\[
y^t g(x) = 0, \quad (12)
\]
\[
g(x) \geq 0, \quad (13)
\]
\[
y_0, y \geq 0, \quad (y_0, y) \neq 0, \quad y_0 \in \mathbb{R}, y \in \mathbb{R}^m, \quad (14)
\]

then \( \bar{x} \) is \( (P) \)-optimal.

Assuming \( f \) to be pseudoconvex at \( \bar{x} \in X \) and assuming \( g_i \) [where \( I = \{ i; g_i(\bar{x}) = 0 \} \)] and \( h \) to be strictly pseudoconcave at \( \bar{x} \in X \), Bector and