Optimal Solutions to Differential Inclusions in Presence of State Constraints

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Abstract. Necessary conditions in terms of the Hamiltonian are given for optimal solutions to the differential inclusion problem when state constraints are present. This result extends a result of Clarke for the unconstrained problem. The data are nonsmooth, nonlinear, nonconvex. The method incorporates the state constraint in the cost functional as a penalty term for a sequence of unconstrained problems that approximate our problem. An application of Ekeland's variational principle, the known necessary conditions for the auxiliary problems, and a limiting process provide the necessary conditions.

Key Words. Differential inclusions, Hamiltonian, state constraints, multifunctions, generalized Jacobian, penalty terms, Ekeland's variational principle.

1. Introduction

In Ref. 1, Clarke derives necessary conditions in terms of the Hamiltonian for the problem

$$\min \{ \theta(x(0)) + \phi(x(1)) : \dot{x}(t) \in E(t, x(t)), \text{a.e., } x(0) \in C_0, x(1) \in C_1 \}. $$

In this paper, we adapt the methods of Clarke to also include state constraints of the form

$$h(t, x(t)) \leq 0,$$

without restricting his hypotheses on the data.

Here, \( E(t, s) \) is a subset of \( \mathbb{R}^n \) for each \((t, s)\), and the problem is known as a differential inclusion problem. It is closely related to the control problem

$$\dot{x}(t) = f(t, x(t), u(t)), \quad u(t) \in U(t), \text{ a.e.,}$$

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since, under appropriate measurability assumptions, it is equivalent to

$$\dot{x}(t) \in E(t, x(t)), \quad \text{with} \ E(t, s) = f(t, s, U(t)).$$

However, there are multifunctions $E$ that have no such representation in terms of $f$ and $U$; the differential inclusion problem subsumes as a special case the optimal control problem (Ref. 1).

Concretely, in the differential inclusion setting, one can assume control constraints that depend on the state,

$$u(t) \in U(t, x(t)).$$

For a more detailed discussion, see Ref. 1, where also an example is given.

On the other hand, results for the differential inclusion problem yield the maximum principle for the optimal control problem (Ref. 1). Our method is closely related to that of Ref. 2. A penalty term is added to a sequence of unconstrained problems which, under a well-posedness assumption (H4 of Section 3), approximate the original state-constrained problem. Ekeland's variational principle (Ref. 3) plays again a central role in our arguments. The proof of the main result and the mathematical machinery developed in Section 4 are related to, but different from, those in Ref. 2.

More concretely, the basic ideas of adding a penalty term and using Ekeland's variational principle are borrowed from Refs. 2 and 7; yet, the auxiliary results are stated and proved in terms of multifunctions. Some of these results may be new. The classical state-constrained control problem with nonsmooth data has been investigated and solved in various papers; for a more informative discussion, see Ref. 2. We then lay stress on the fact that we solve a problem that subsumes as a special case the conventional control problem and directly extends the results of Ref. 1 to the state-constrained problem. All results are stated in terms of generalized gradients, as no differentiability or convexity assumptions are imposed.

2. Notation

We denote by $I$ the unit interval $[0, 1]$ and by $E$ a finite-dimensional Euclidean space. $\mathcal{L}$ will denote the $\sigma$-algebra of Lebesgue subsets of $I$ and $\mathcal{B}(E)$ the $\sigma$-algebra of Borel subsets of $E$. The terms measurable and almost everywhere (a.e.) are understood with respect to Lebesgue measure.

$\mathcal{N}(E)$ will denote the set of normalized functions of bounded variation defined on $I$ and with values in $E$, that is, functions of bounded variation which are left continuous on $(0, 1)$.

Signed, Radon measures on $\mathcal{B}(I)$ will be called measures. The class of positive measures will be denoted by $C^\oplus(I)$. 