TECHNICAL NOTE

On Optimal Periodic Control and Nested Optimization Problems

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Abstract. A minimization problem for a functional on a convex subset \( C \) of a normed linear space is considered. Under certain hypotheses, optimality in a certain subset of \( C \) implies the validity of first-order necessary optimality conditions for the problem in \( C \). The result is applied to a problem in optimal periodic control of neutral functional differential equations.

Key Words. Optimal periodic control, first-order optimality conditions, neutral functional differential equations.

1. Introduction

This note is motivated by optimal periodic control problems. Here, a fundamental problem is to discern optimal steady-state solutions which are merely optimal among steady states from those which are also optimal among periodic solutions (see e.g. Ref. 1 and Section 3 below). For systems governed by ordinary differential equations, it is well known that first-order necessary optimality conditions for optimality in the (restricted) class of steady states coincide with the corresponding conditions for optimality in the larger class of periodic solutions. This excited interest in higher-order optimality conditions (see, e.g., Refs. 1–3).

In Ref. 4, it was shown that a similar result holds for optimal periodic control of retarded functional differential equations by deriving the concrete
form of the optimality conditions. This, however, required a lot more effort and induced the question of which particular features of the underlying optimization problem(s) are responsible for this phenomenon.

Section 2 gives a simple characterization of two nested optimization problems leading to identical first-order optimality conditions. Section 3, illustrates that this abstract result captures in fact the relevant features of optimal periodic control problems. Here, we deal with neutral functional differential equations (including, in particular, the standard optimal periodic control problem for ordinary differential equations). Functional differential equations are of interest in this context, since, in control problems for chemical reactions, delays occur frequently (e.g., due to recycle loops). These problems, besides flight performance optimization, are a major field of applications for optimal periodic control.

2. Nested Optimization Problems

Consider the following optimization problem:

\[(P) \quad \text{minimize } g(y), \quad \text{s.t. } y \in C \subseteq Y;\]

here, \(C\) is an open subset of the normed linear space \(Y\), \(g: C \to \mathbb{R}\) has a Gateaux derivative \(g'(y_0, y)\) at \(y_0 \in C\) in direction \(y \in Y\), and \(C\) is a closed and convex subset of \(Y\). A first-order necessary optimality condition for a local minimum \(y_0 \in C\) has the form

\[g'(y_0, y - y_0) \geq 0,\]

for all \(y \in C\). As is well known, there are many problems where this condition is not only satisfied by local minima, but by other points \(y_0\), too. Most frequently, this occurs when \(g'(y_0, \cdot) = 0\). In such a case, one has to seek recourse to either higher-order necessary optimality conditions or to sufficient optimality conditions (see Ref. 5).

Below, we describe a situation, where points \(y_0\) which are only optimal with respect to a certain subset \(\tilde{C}\) of \(C\) satisfy condition (1) not only for elements \(y\) in \(\tilde{C}\), but for all elements \(y\) in \(C\).

Thus, we consider, in addition to the problem \((P)\) formulated above, the following minimization problem, sitting inside \((P)\):

\[(\tilde{P}) \quad \text{minimize } g(y), \quad \text{s.t. } y \in \tilde{C},\]

where \(\tilde{C}\) is a subset of \(C\). We have the following result.

**Proposition 2.1.** Let \(y_0 \in \tilde{C}\) satisfy the first-order optimality condition for \((\tilde{P})\); i.e., suppose that (1) holds for all \(y \in \tilde{C}\). Assume that there exists