Optimal Control for Nonlinear Systems Calculated with Small Computers

B. Aselmeyer

Communicated by R. Sargent

Abstract. Using Ritz's procedure of representing the control functions of an optimal control problem by a function series with parameters to be optimized, it is shown that, from the well-known gradient procedure for dynamic problems, a simple iteration formula for the optimization of these parameters can be derived. Using an example with a technical background, the effectiveness of the program realization of this approach is demonstrated and is compared with the results of unrestricted dynamic optimization.

Key Words. Dynamic optimization, Ritz parametrization of the control functions, nonlinear systems, nonlinear programming, projection method.

1. Introduction

For technical applications, many optimization problems can be formulated in this way: Find the minimum of a functional $P$,

$$P(x(t_f), t_f) \rightarrow \min$$  \hspace{1cm} (1)

$(x: n \times 1$ vector), where the state vector $x$ is defined by a nonlinear system of differential equations of first order,

$$\dot{x} = g(x, u, t)$$  \hspace{1cm} (2)

$(g: n \times 1$ function vector, $u: l \times 1$ vector), by choosing an appropriate control vector $u$. The elements of $u$ may be bounded,$^3$

$$(u_i)_{\text{min}} \leq u_i \leq (u_i)_{\text{max}}.$$  \hspace{1cm} (3)

---

$^1$ This work was performed at the Technische Hochschule in Darmstadt, West Germany, with financial support from the DFG (Deutsche Forschungs-Gemeinschaft).

$^2$ Staff Member, Development Center, R. Bosch GmbH, Stuttgart, West Germany.

$^3$ More general boundaries for $u$ are possible (for instance, $u$ is an element of a convex region in the control space), but are not considered here. This is due to the fact that the boundaries for single elements in $u$ correspond in technical systems to limitations in the controlling elements.
The initial condition is given by
\[ x(t_0) = a \]  \hspace{1cm} (4)
\((a: n \times 1 \text{ constant vector})\), and the final condition is given by
\[ q(x(t_i), t_f) = 0 \]  \hspace{1cm} (5)
\((q: r \times 1 \text{ function vector with } r \leq n)\), where the elements of \( q \) are also nonlinear functions. The calculus of variations provides the theoretical background for the solution of this problem. The approaches of Euler-Lagrange and Pontryagin yield feasible numerical methods, which can be applied also to problems of realistic order; however, the computational effort is quite considerable. If optimal control problems are to be solved on small computers [i.e., computers with restricted memory and relatively slow execution speed, like personal computers, desktop calculators, or mini/micro computers], these procedures must be simplified.

The approach taken in this paper is based on an idea of Ritz (Ref. 1), who first presented direct methods in the calculus of variations. For the problem stated above, a parametrization of the control functions is introduced,
\[ u_i(t) = \sum_{j=1}^{m_i} p_{ij} f_{ij}(t), \]  \hspace{1cm} (6)
where \( f_{ij}(t) \) are suitably chosen, known functions of the time and \( p_{ij} \) are parameters combining these basis functions. A similar approach has been adopted in Refs. 2–5, using either direct search methods (Refs. 2 and 3) or gradient methods (Refs. 4 and 5). Convergence properties of the control parametrization Ritz method have been studied in Ref. 6.

In this paper, the Ritz parametrization method will be used in the derivation of an iteration formula based on Kelley's approach (Ref. 8) to the steepest-descent method. As a result, the dynamic optimization problem is reduced to a parameter optimization problem, where the iteration formula is the one known from nonlinear programming. Other approaches (e.g., the approach used by Bryson et al., Ref. 7) yield the same result and are discussed in Ref. 11. To demonstrate that this simplification allows the solution of technically relevant problems on small computers, the problem of bringing the upper stage of a rocket into a prescribed orbit is solved using a program based on the method presented here; this was executed on a desktop calculator. Some simulations are presented in Section 4.

Of course, it is clear that the results obtained from restricting \( u_i \) according (6) to a function series are only suboptimal, in the sense that a solution is found only for the given basis functions; in addition, the arbitrary choice of the number of parameters as well as the type of basis functions