Polyhedral Functions and Multiparametric Linear Programming

M. SCHECHTER

Communicated by P. L. Yu

Abstract. In a linear programming problem with a vector parameter appearing on the right-hand side, the minimum value of the objective is a polyhedral function of this parameter. We show how different characterizations of a polyhedral function correspond to different ways of solving the right-hand side multiparametric linear programming problem.

Key Words. Polyhedra, convexity, multiparametric linear programming.

1. Introduction

Consider the linear multiparametric programming problem

\[ \begin{align*}
\min \ z &= c^T x, \\
\text{s.t.} \ A x &= b + F \lambda, \quad x \geq 0,
\end{align*} \]

where \( A \) is an \( m \times n \) matrix, \( F \) is an \( m \times k \) matrix, and \( x, c, \lambda \) are column vectors of the appropriate dimensions. The minimum value of the objective function is a polyhedral function \( z^*(\lambda) \). By the method of Gal and Nedoma (Refs. 1 and 2), it is found by computing a number of polyhedra in the \( \lambda \) space such that \( z^* \) is affine on each of these polyhedra. On the other hand, \( z^* \) can be computed as the maximum of the dual problem (as has been implicitly done by Hailperin in Ref. 3). In this form, \( z^* \) is naturally represented as the maximum of a finite set of affine functions. These are two forms in which any polyhedral function can be represented. We study in detail the relationships between these two representations and apply the results to the multiparametric linear programming problem.

\[ ^1 \text{Professor, Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania.} \]
2. Unions of Convex Sets

This section presents results which will be needed in the analysis of polyhedral functions. All sets which occur here, as in the rest of the paper, are subsets of \( R^n \).

The first lemma may be proved by an elementary argument or may be deduced from the fact that a hyperplane in \( R^n \) has Lebesgue measure zero.

**Lemma 2.1.** A set which is the union of finitely many hyperplanes has empty interior.

In the next theorem, the set \( C \) is assumed to have nonempty interior. This is merely for convenience; by referring to the relative topology of the affine hull of \( C \), this assumption could be discarded.

**Theorem 2.1.** Let \( C = \bigcup \{ C_i, i = 1, \ldots, m \} \), where each \( C_i \) is a closed convex set. If \( C \) is convex and has nonempty interior, then

\[
C = \bigcup \{ C_i | \text{int } C_i \neq \emptyset \}.
\]

**Proof.** Let \( I = \{ i | \text{int } C_i \neq \emptyset \} \), and let

\[
C^0 = \bigcup \{ C_i, i \in I \},
\]

\[
C^1 = \bigcup \{ C_i | i \notin I \}.
\]

Then,

\[
C = C^0 \cup C^1.
\]

Let \( x^* \in \text{int } C \). Suppose that \( x^* \notin C^0 \). Since the latter is a closed set, some neighborhood of \( x^* \) is disjoint from \( C^0 \) and lies in \( C \), therefore lies in \( C^1 \); but, for each \( i \notin I \), \( C_i \) lies in a hyperplane; therefore, by Lemma 2.1, \( C^1 \) contains no neighborhood, hence \( x^* \in C^0 \), so that \( \text{int } C \subseteq C^0 \). Since \( C \) is convex and closed,

\[
C = \text{cl}(\text{int } C) \subseteq \text{cl}(C^0) = C^0,
\]

therefore, \( C = C^0 \).

**Theorem 2.2.** Let \( P_1, \ldots, P_n \) be polyhedra whose union is convex. Then, the union is a polyhedron.

**Proof.** For any set \( A \), let \( \text{conv } A \) denote the convex hull of \( A \), and let \( \text{cone } A \) denote the conical hull of \( A \), i.e.,

\[
\text{cone } A = \{ \sum \mu_i a_i | \mu_i \geq 0, a_i \in A \}.
\]