Optimal Projection Equations
for Reduced-Order Modelling, Estimation, and Control
of Linear Systems with Multiplicative White Noise\textsuperscript{1,2}

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Abstract. The optimal projection equations for quadratically optimal reduced-order modelling, estimation, and control are generalized to include the effects of state, control, and measurement dependent noise.

Key Words. Feedback control, robust control, fixed-order compensation, optimal control.

1. Introduction

As is well known, LQR and LQG controllers lack guaranteed robustness with respect to arbitrary parameter variations (Refs. 1 and 2). A widely studied approach to correcting this defect involves introducing noise into the plant via the imperfectly known parameters (Refs. 3-10). Intuitively speaking, the quadratically optimal feedback controller designed in the presence of such disturbances is automatically desensitized to actual parameter variations. This was demonstrated in Ref. 11 for the example given in Ref. 1.

The contribution of the present paper is a generalization of classical steady-state LQG theory to include the effects of state, control, and measurement dependent noise. In contrast to the classical solution involving a pair

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of separated Riccati equations, the necessary conditions for quadratic optimality in the presence of multiplicative white noise consist of a system of two modified Riccati equations and two modified Lyapunov equations coupled by stochastic effects. The coupling serves as a graphic portrayal of the breakdown of the separation principle in the multiplicative noise case. When the multiplicative noise terms are set to zero, the modified Lyapunov equations drop out and the modified Riccati equations immediately reduce to the standard pair of separated LQG Riccati equations. Related results were obtained for the discrete-time, finite-interval problem in Ref. 10.

To attain further generality, a constraint is imposed on controller order as in Ref. 12. Hence, the results of the present paper also constitute a direct generalization of the coupled system of modified Riccati and Lyapunov equations which arise in characterizing reduced-order controllers.

For the special case of full-order compensation in the presence of state-dependent noise only, versions of these equations were discovered independently by Hyland (Refs. 13 and 14) and Mil'stein (Ref. 15). An interesting difference between Refs. 13–14 and Ref. 15 is that Mil'stein interpreted the plant model as an Ito stochastic differential equation, whereas Hyland utilized the Fisk–Stratonovich definition (Refs. 16–18). In earlier work on modelling flexible mechanical structures (Refs. 19 and 20), justification for this interpretation as an appropriate model for parameter uncertainty was based upon the maximum entropy principle of Jaynes (Ref. 21) and the theory of stochastic approximation (Ref. 22). A summary of this approach and its relationship to Refs. 23 and 24 can be found in Ref. 25. Rigorous guarantees of robustness over a prescribed range of parameter variations have been obtained using Lyapunov functions (Refs. 26–29). Although the present paper utilizes an Ito model for simplicity, results based on Stratonovich models are readily obtained by means of standard transformations.

An immediate practical benefit of the structured form of the necessary conditions is the means for constructing numerical algorithms which differ fundamentally from gradient search techniques. One such iterative algorithm, proposed in Refs. 30–32, exploits the characterization of the oblique projection as the sum of rank-1 eigenprojections of the product of the rank-deficient pseudogramians satisfying the modified Lyapunov equations. As discussed in Ref. 32, the necessary conditions fail to specify which eigenprojections comprise the oblique projection; indeed, each choice may correspond to a local extremal. In practice, judicious choice of the eigenprojections can eliminate extremals with high cost and hence efficiently identify the global minimum. These issues are a result of the reduced-order constraint only; the stochastic effects alone do not appear to introduce extremal multiplicity.