Games of Timing with Resources of Mixed Type

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Abstract. The paper considers a game of timing which is closely related to the so-called duels. This is a game connected with the distribution of resources by two players. Each of the players is in possession of some amount of resource to be distributed by him in the time interval \([0, 1]\). In his behavior, Player 1 is restricted by the necessity of taking all of his resources at a single point, while Player 2 has no restrictions. For the payoff function, defined as for duels, the game is solved; explicit formulas on the value of the game and the optimal strategies for the players are found.

Key Words. Games of timing, duels, distribution of resources.

1. Introduction

In this paper, we consider some general zero-sum two-person game of timing, of the duel type, which will be described as games related to distribution of resources. More precisely, we consider the following auxiliary model, whose idea has been taken from Ref. 1, Section 2.4. A player possesses an amount \(M > 0\) of resource to be distributed by him in the time interval \([0, 1]\), according to a measure \(\mu\), with \(\mu[0, 1] = M\). We assume that the player, using any strategy \(\mu\), achieves success in each subinterval \(D\) of \([0, 1]\) with a probability which depends on only the restriction of \(\mu\) to the set \(D\). Let \(Q^\mu(D)\) and \(Q^\mu(t)\) denote the probabilities that he succeeds in an interval \(D\) or in \([0, t]\), respectively, after using \(\mu\). From the definition of the model,

\[
Q^\mu[t, t + h] = \mu[t, t + h] \cdot A(t) + o(h), \quad \text{a.e., } 0 \leq t < 1, \tag{1}
\]

for any absolutely continuous measure \(\mu\) on \([0, 1]\), with \(\mu[0, 1] = M\), where \(A(t)\), the so-called modified accuracy function, is a continuous and non-decreasing function from \([0, 1]\) onto \([c, d]\), \(0 \leq c < d \leq \infty\).

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Equation (1) implies (see Ref. 1, Section 2.4) the following formula:

\[ Q^\mu(t) = 1 - \exp\left\{ - \int_{[0,t]} A d\mu \right\}, \quad 0 \leq t \leq 1, \quad (2) \]

for any absolutely continuous measure \( \mu \).

Now, we can extend (2) on the set of all measures \( \mu \) on \([0, 1]\). However, this model is deficient in the following sense: for example, if we substitute in (1) \( h \to 0^+ \) and \( \tilde{\mu} = MI_t \), as the measure placing the total mass \( M \) at the point \( t \), then we get

\[ Q^\tilde{\mu}(t) = MA(t), \]

contradicting (2).

The model considered above can be improved and generalized in the manner given below, which makes it strongly related to the classical discrete duels. The construction is a modification of the one given in Ref. 2.

Let us call the function

\[ \tilde{Q}(t) = Q^I[t, t], \quad 0 \leq t \leq 1, \]

the accuracy function of the player, where \( I_t \) denotes the probability measure placing its total mass at the point \( t \). The value \( \tilde{Q}(t) \) can be interpreted (as in discrete duels) as the probability of success of the player at the point \( t \) when he takes the resource of amount 1 at this point. In general, we require \( \tilde{Q}(t) \) to be a measurable function from \([0, 1]\) into \([0, 1]\). Now, after defining

\[ \tilde{Q}^\mu(D) = 1 - Q^\mu(D), \]

the following conditions seem reasonable:

(i) for any measurable set \( D \subseteq [0, 1] \) and \( \alpha \geq 0, \)

\[ 0 \leq \tilde{Q}^{\alpha I_t}(D) \leq 1, \quad \tilde{Q}^I(D) = \tilde{Q}(t), \quad \text{if } t \in D; \]

(ii) for any measurable set \( D \subseteq [0, 1] \) and for all \( \alpha \geq 0, \beta \geq 0, \)

\[ \tilde{Q}^{(\alpha + \beta)I_t}(D) = \tilde{Q}^{\alpha I_t}(D) \cdot \tilde{Q}^{\beta I_t}(D), \quad \text{if } t \in D; \]

(iii) for any measure \( \mu \) on \([0, 1]\) and for all nonempty measurable sets \( D \subseteq [0, 1], \)

\[ \inf_{t \in D} \tilde{Q}^{\alpha I_t}(D) \leq \tilde{Q}^\mu(D) \leq \sup_{t \in D} \tilde{Q}^{\alpha I_t}(D), \]

where \( \alpha = \mu(D) \);

(iv) for any sequence \( \{D_i\} \) of disjoint measurable subsets of \([0, 1]\), and for any measure \( \mu \) on \([0, 1], \)

\[ \tilde{Q}^\mu\left( \bigcup_i D_i \right) = \prod_i \tilde{Q}^\mu(D_i). \]