Implementing Proximal Point Methods for Linear Programming

S. J. Wright

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Abstract. We describe the application of proximal point methods to the linear programming problem. Two basic methods are discussed. The first, which has been investigated by Mangasarian and others, is essentially the well-known method of multipliers. This approach gives rise at each iteration to a weakly convex quadratic program which may be solved inexactly using a point-SOR technique. The second approach is based on the proximal method of multipliers, originally proposed by Rockafellar, for which the quadratic program at each iteration is strongly convex. A number of techniques are used to solve this subproblem, the most promising of which appears to be a two-metric gradient-projection approach. Convergence results are given, and some numerical experience is reported.

Key Words. Linear programming, method of multipliers, convex quadratic programming.

1. Introduction

We consider the general programming problem

$$\min_{z} g^T z, \quad \text{s.t. } Bz = d, \ z_j \geq 0, \ j = 1, \ldots, q,$$

(1)
where \( z, g \in R^n, d \in R^n, B \in R^{m \times n}, 0 \leq q \leq n \). The usual formulations (standard form, inequality-constrained form, and their respective duals) can easily be expressed as (1) after appropriate use of slack variables, and the algorithms to be described in Sections 2 and 3 can then be applied. Throughout the paper, we make the following assumptions.

**Assumption 1.1.** The linear program (1) has a finite optimal solution.

**Assumption 1.2.** The matrix \( B \) has no zero columns.

In this paper, we describe the use of two methods, the method of multipliers and the proximal method of multipliers, to solve (1). Both of these methods are shown by Rockafellar (Ref. 1) to be derivable from the proximal point algorithm, the theory for which is developed in Refs. 1 and 2. Essentially, the proximal point algorithm solves the general convex programming problem

\[
\min_{x \in S} f(x), \quad S \subset R^n \text{ convex and closed}, \quad f: R^n \to R \text{ convex},
\]

by generating a sequence \( x^i \) as follows:

\[
x^{i+1} = \arg \min_{x \in S} f(x) + (\gamma_i/2)\|x - x^i\|^2,
\]  

(2)

where \( \{\gamma_i\} \) is some sequence with \( \gamma_i \downarrow \gamma_\infty \geq 0 \). Here and throughout the paper, \( \| \cdot \| \) is used to denote the Euclidean norm. Clearly, it is not desirable to apply this directly to (1), since the subproblem (2) would probably be more difficult to solve than the original problem. However, in Ref. 1, Rockafellar shows that the method of multipliers can be obtained by applying (2) to the dual of (1). If the primal feasible set \( C \) is defined by

\[
C = \{z \in R^n \mid z_j \geq 0, \quad j = 1, \ldots, q\}
\]

and the augmented Lagrangian for (1) as

\[
L(z, y, \gamma) = g^Tz + y^T(Bz - d) + (1/2\gamma)\|Bz - d\|^2,
\]  

(3)

the exact form of the algorithm consists of successive minimizations of \( L(\cdot, y^i, \gamma_i) \), followed by updates of the Lagrange multipliers,

\[
y^{i+1} \leftarrow y^i + (1/\gamma_i)(Bz^{i+1} - d).
\]  

(4)

Clearly, the main computational step in this algorithm is the repeated minimization of (3) with respect to \( z \) on the set \( C \), which is a bound-constrained convex quadratic programming problem. It is easy to show,