Optimality for Set Functions with Values in Ordered Vector Spaces\textsuperscript{1,2}

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Abstract. Let \((X, \Gamma, \mu)\) be a finite atomless measure space, \(\mathcal{F}\) a convex subfamily of \(\Gamma\), and \(Y\) and \(Z\) locally convex Hausdorff topological vector spaces which are ordered by the cones \(C\) and \(D\), respectively. Let \(F: \mathcal{F} \to Y\) be \(C\)-convex and \(G: \mathcal{F} \to Z\) be \(D\)-convex set functions. Consider the following optimization problem \((P)\): minimize \(F(\Omega)\), subject to \(\Omega \in \mathcal{F}\) and \(G(\Omega) \leq_D \theta\). The paper generalizes the Moreau-Rockafellar theorem with set functions. By applying this theorem, a Kuhn–Tucker type optimality condition and a Fritz John type optimality condition for problem \((P)\) are established. The duality theorem for problem \((P)\) is also studied.

Key Words. Convex set functions, strictly convex set functions, convex subfamily of measurable subsets, ordered vector spaces, normal cones, \(C\)-convex set functions, minimal points, weak minimal points, saddle points, weak saddle points, subdifferentials, weak subdifferentials, order-complete vector lattices.

1. Introduction

Throughout the paper, let \((X, \Gamma, \mu)\) be an atomless finite measure space with \(L_1(X, \Gamma, \mu)\) separable and \(\mathcal{F}\) a convex subfamily of \(\Gamma\). Let \(Y\) and \(Z\) be locally convex, Hausdorff real topological vector spaces. Assume further

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that $Y$ and $Z$ are ordered complete vector lattices ordered by the closed convex pointed cones $C$ and $D$, respectively. We consider the following optimization problem:

$$\text{(P) minimize } F(\Omega),$$
$$\text{subject to } \Omega \in \mathcal{S} \text{ and } G(\Omega) \leq_D 0,$$

where $F: \mathcal{S} \rightarrow Y$ and $G: \mathcal{S} \rightarrow Z$ are respectively $C$-convex and $D$-convex set functions, and $0$ is the zero vector. This type of optimization problem involves a variety of interesting applications (see Ref. 1), yet the theory is not adequately cited in usual optimization. Recently, there have been many authors who have studied the optimization problem for set functions; see Refs. 1-10. In particular, see the initial work of Morris (Ref 1); the range of set functions are mostly taken in $\mathbb{R}$ or $\mathbb{R}^n$. In Ref. 6, the author first considers the set functions with values in ordered topological vector spaces. In Section 3, we generalize the Moreau-Rockafellar type theorem for set functions whose values are in $Y$. We consider the weak subdifferential $\partial_w$ for set functions and prove that

$$\partial_w(F_1 + F_2)(\Omega) \subseteq \partial_w F_1(\Omega) + \partial_w F_2(\Omega), \quad \Omega \in \Gamma,$$  

(1)

holds, but not the reverse inclusion.

A counterexample shows that the equality relation in (1) is not valid, except when $Y = \mathbb{R}$. The fact that the equality holds in (1) for $Y = \mathbb{R}$ was shown in Theorem 6 of Lai and Lin (Ref. 7). If the weak subdifferential is replaced by the subdifferential, then it can be shown that a Moreau-Rockafellar type theorem is also true for set functions with values in order-complete vector lattice locally convex space. That is,

$$\partial (F_1 + F_2)(\Omega) = \partial F_1(\Omega) + \partial F_2(\Omega), \quad \Omega \in \Gamma.$$

Using the results obtained in Section 3, we show a generalized Kuhn-Tucker type theorem and a Fritz John type theorem in Section 4. In particular, when $Y = Z = \mathbb{R}$, the generalized Kuhn-Tucker type theorem is reduced to Theorem 11 of Lai and Lin (Ref. 7); and the generalized Fritz John type theorem is similar to Theorem 12 of Lai and Lin (Ref. 7). Many authors (see, for example, Refs. 11-14) studied the duality theorems for usual functions on linear spaces, rather than on the $\sigma$-algebra $\Gamma$. As an application of the generalized Kuhn-Tucker type theorem, we investigate a duality theorem for set functions in Section 5. For completeness, we proceed with some basic concepts and definitions for set functions. Most notations and definitions will follow Ref. 6; see also Refs. 1 and 7.